A DUALITY THEOREM FOR THE K-GROUPS OF HOMOLOGY AND COHOMOLOGY

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Abstract

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A DUALITY THEOREM FOR THE K-GROUPS OF HOMOLOGY AND COHOMOLOGY

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In the note ([1]) a definition was given of the p-dimensional K-groups of homology and cohomology of a space A over a coefficient group G. In the present note we shall retain the definition of the p-dimensional K-group of homology $\Delta_p^K(A, G)$, while the definition of the K-groups of cohomology will be somewhat modified. Namely, if L is an arbitrary locally finite complex, then by a p-dimensional K-cochain in L over the group G is meant such a collection of cochains $y = \{y^\sigma\}_{\sigma \in K}$ of the complex L over the group G that $\dim y^\sigma = p - \dim \sigma$. Further, the K-coboundary operator δ is defined by the formula $(\delta y)^\sigma = \delta(y^\sigma) + (-1)^{\dim y^{\sigma - 1}} y_{\delta\sigma}$. These formulas differ from the formulas for K-cohomology in ([1]) only by signs in the corresponding places. In all other respects the construction of the K-groups of cohomology remains the same as in ([1]). Everywhere in what follows cohomology is understood in the sense modified here.

Duality theorem. *Let M^n be a closed oriented n-dimensional homological manifold; let p, q be nonnegative integers whose sum is $n-1$, i.e. $p+q = n-1$. Suppose in addition that M^n is K-acyclic in dimensions q and $q+1$ with respect to the coefficient group G. Then for any set $A \subset M^n$ and $B = M^n \setminus A$ it turns out that the K-groups of cohomology $\nabla_K^p(A, G)$ and of homology $\Delta_q^K(B, G)$, taken over the coefficient group G, are isomorphic to one another:*

$$\nabla_K^p(A, G) \simeq \Delta_q^K(B, G).$$

For the proof of this theorem we shall need to define the external K-groups of homology and cohomology $\Delta_{pBH}^K(A, G)$ and $\nabla_{KBH}^p(A, G)$. Let $\{\tau\}$ be the directed set of all triangulations τ of all possible neighborhoods of the set $A \subset M^n$. For any triangulation τ of some neighborhood λ of the set A in M^n, consider the K-group of homology $\Delta_p^K(\tau, G)$ (see ([1])). Now let the triangulation τ' follow the triangulation τ (see ([2])), and let an arbitrary K-chain $x_p = \{x_\sigma\}$, $\sigma \in K$, of the triangulation τ be given. We then define the K-chain $S_{\tau'}^\tau x_p$ of the triangulation τ' as follows. Take a simplex t' of the triangulation τ' and its carrier t in τ. If the dimensions of the simplexes t' and t coincide, then we put $S_{\tau'}^\tau x_p(t') = x_p(t)$; otherwise we put $S_{\tau'}^\tau x_p(t') = 0$. It is easily checked that the equality

$$\partial S^{\tau}_{\tau'} x_p = S^{\tau}_{\tau'} \partial x_p$$

holds. Thus, the operator $S^{\tau}_{\tau'}$ gives rise to a homomorphism $S^{\tau}_{\tau'} : \Delta^K_p(\tau, G) \to \Delta^K_p(\tau', G)$. These groups and homomorphisms define a direct spectrum $\{\Delta^K_p(\tau, G);\, S^{\tau}_{\tau'}\}$. We denote the limiting group of this spectrum by $\Delta^K_{pBH}(A, G)$ and, by definition, call it the external K-group of homology of dimension p of the set A over the coefficient group G.

We now define the external K-group of cohomology. Consider again the directed system $\{\tau\}$ of all triangulations τ of all possible neighborhoods of the set $A \subset M^n$. Let τ' follow τ and let φ be some canonical displacement of the triangulation τ' into τ. Then, by a δ-subdivision of an arbitraryly

taken K-cochain $y^p = \{y^\sigma\}$ of the triangulation τ we shall call the K-cochain $\bar{\varphi} y^p$ of the triangulation τ', defined by the equality

$$\bar{\varphi} y^p(t') = y^p(\varphi(t')),$$

where $t' \in \tau'$ and $\varphi(t') \in \tau$. It is easily shown that the operator $\bar{\varphi}$ commutes with the K-coboundary operator δ. Hence one obtains a homomorphism $\bar{\varphi}_{\tau,\tau'} : \Delta^p_K(\tau, G) \to \nabla^p_K(\tau', G)$. These groups and this homomorphism determine a direct spectrum $\{\nabla^p_K(\tau, G); \bar{\varphi}_{\tau,\tau'}\}$. The limit group of this spectrum will be denoted by $\nabla^p_{KBH}(A, G)$ and will be called, by definition, the external K-group of cohomology of dimension p of the set A over the coefficient group G.

The proof of the duality theorem formulated above reduces to the proof of the following three isomorphisms:

$$f : \nabla^p_{KBH}(A, G) \approx \nabla^p_K(A, G),$$

$$D : \nabla^p_{KBH}(A, G) \approx \Delta^K_{(n-p)BH}(A, G),$$

$$\varphi : \Delta^K_{(n-p)BH}(A, G) \approx \Delta^K_q(B, G).$$

The composition of these isomorphisms gives the required isomorphism

$$M = \varphi D f^{-1} : \nabla^p_K(A, G) \approx \Delta^K_q(B, G).$$

Let us briefly outline the construction of these three isomorphisms.

The construction of the isomorphism $f : \nabla^p_{KBH}(A, G) \approx \nabla^p_K(A, G)$ is carried out as follows. Let $y^p = \{y^\sigma\}$ be an arbitrary external K-cocycle of the set A, lying on a triangulation τ of some neighborhood λ of the set A. To the triangulation τ we assign a covering ω_τ of the set A, consisting of the sets cut out from A by the stars of the vertices of the triangulation τ. The nerve ω_τ of the covering ω_τ is a certain closed subcomplex of the complex τ. Next, by fy^p we denote the K-cocycle of the nerve ω_τ which takes on each simplex of the complex ω_τ the value that the K-cocycle y^p took on this simplex. As a result, to each external K-cocycle y^p of the set A there is assigned a definite K-cocycle fy^p of the set A. One can show that this correspondence establishes an isomorphism

$$f : \nabla^p_{KBH}(A, G) \approx \nabla^p_K(A, G).$$

The construction of the isomorphism $D : \nabla^p_{KBH}(A, G) \approx \Delta^K_{(n-p)BH}(A, G)$ proceeds as follows. Let τ be a triangulation of some neighborhood λ of the set A, and let $C^p_K(\tau, G)$ be the group of K-cochains of dimension p of the triangulation τ over G. Further, let $y^p = \{y^\sigma\} \in C^p_K(\tau, G)$. For the K-cochain y^p we define the K-chain $D_{q+1}(y^p)$ of dimension $q + 1$ of the star complex conjugate to τ, τ^* (see (2)), requiring, as usual, that $D_{q+1}(y^p)$ take on the conjugate barycentric star $v_i^{q+1+\dim\sigma} \in \tau^*$ of the simplex $t_i^{p-\dim\sigma} \in \tau$ the value that y^p takes on $t_i^{p-\dim\sigma}$:

$$\left(D_{q+1}(y^p)\, v_i^{q+1+\dim\sigma}\right) = y^p\left(t_i^{p-\dim\sigma}\right).$$

This correspondence defines an isomorphism $C^p_K(\tau, G) \approx C^K_{q+1}(\tau^*, G)$. It is easily shown that

$$\partial D_{q+1}(y^p) = (-1)^{p+1} D_q \delta(y^p).$$

From this equality one obtains the isomorphism:

$$D_{q+1} : \nabla^p_K(\tau, G) \approx \Delta^K_{q+1}(\tau^*, G).$$

The star K-chain $D_{q+1}(y^p) = c^*_{q+1}$ can be regarded as a K-chain of the barycentric subdivision τ_1 of the triangulation τ, and, subjecting the barycentric subdivision τ_1 to the canonical shift into τ, we transfer

c^*_{q+1} into a K-chain c_{q+1} of the complex τ. It is easy to show that this correspondence defines an isomorphism

$$D_1 : \Delta^K_{q+1}(\tau^*, G) \simeq \Delta^K_{q+1}(\tau, G).$$

The isomorphism D_1, composed with the isomorphism D_{q+1}, gives the isomorphism

$$\Delta^p_K(\tau, G) \simeq \Delta^K_{q+1}(\tau, G).$$

To complete the construction of the isomorphism

$$D : \nabla^p_{KBH}(A, G) \simeq \Delta^K_{(q+1)BH}(A, G)$$

the following lemma, proved in ([1]), is used.

Lemma. If $\tau' > \tau$, then

$$SDY^p \sim DSY^p,$$

where S denotes on the left the operator of ∂-subdivision, and on the right—the operator of δ-subdivision from τ to τ'.

Finally, the construction of the isomorphism

$$\varphi : \Delta^K_{(q+1)BH}(A, G) \simeq \Delta^K_q(B, G)$$

proceeds according to the following scheme. Represent the triangulation τ of a neighborhood λ of the set A, on which lies some exterior K-cycle $Z_{q+1} = \{Z_\sigma\}$ of the set A, as a union of closed finite subcomplexes O_k, $k = 1, 2, ...$, and it is assumed that O_k is contained in the open core of the complex O_{k+1}. Next, to each simplex $\sigma_i \in K$ there is assigned a sequence $\sigma_1, \sigma_2, \dots, \sigma_k, ...$ of simplexes of this same complex, "going to infinity." If $z_\sigma \in Z_{q+1}$, then the piece of this K-cycle on the subcomplex O_k of the triangulation τ is denoted by z_{σ_k}; the K-boundary ∂z_{σ_k} of this piece will be denoted by x_{σ_k}. The set of all cycles x_{σ_k} will be denoted by x_q, i.e. set $x_q = \{x_{\sigma_k}\}$. Clearly, the cycles x_{σ_k} lie in λ, but the distances of all their vertices from the compact set $\Phi = M^n \setminus \lambda \subseteq B$ tend to zero as k increases. Therefore, by assigning to each vertex of the cycle x_{σ_k} the nearest point of the compact set Φ, we obtain an infinitely small displacement of the K-cycle $x_q = \{x_{\sigma_k}\}$ into some K-cycle of the compact set $\Phi \subseteq B$, which we again denote by $x_q = \{x_{\sigma_k}\}$. It is clear that the mesh of the chains x_{σ_k} decreases without bound when σ_k "goes to infinity" in the complex K. The correspondence $\varphi : Z_{q+1} \to x_q$, as can be shown, also establishes the required isomorphism

$$\varphi : \Delta^K_{(q+1)BH}(A, G) \simeq \Delta^K_q(B, G).$$

This completes the proof of the duality theorem.

From this theorem, as special cases, many previously known duality relations between homology and cohomology groups are obtained. For example, from it follows the general duality law of P. S. Aleksandrov for arbitrary mutually complementary sets in n-dimensional spherical space (see ([3])), the duality law of K. A. Sitnikov (see ([2])), etc.

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Note: Figure translations are in progress. See original paper for figures.