IN THE THEORY OF UNIVALENT FUNCTIONS

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 517.54

MATHEMATICS

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L. S. PONTRYAGIN' S MAXIMUM PRINCIPLE

IN THE THEORY OF UNIVALENT FUNCTIONS

(Presented by Academician M. A. Lavrent' ev on 27 I 1969)

In this note a method is proposed for solving extremal problems in the theory of univalent functions, based on L. S. Pontryagin' s maximum principle ([1,2]). The method is demonstrated on the problem:

A. Find $\inf \operatorname{Re}\{G(f(z))\}_n$ in the class S of all holomorphic univalent functions $f(z)$ in the disk $|z|<1$, $f(0)=0$, $f'(0)=1$*, for a fixed function $G(z)$, regular at the origin, and a given natural number n.

Relying on the Löwner equation (1°), in 2° we indicate a device that makes it possible to reduce problem A to a quasilinear problem B of optimal control. In 3° the maximum principle is formulated for problem B; in 4° the functions $v(t)$ satisfying the maximum principle are analyzed; in 5° an invariant is obtained for problem B, and in conclusion it is indicated that, under simple initial conditions for the auxiliary variables in problem B, optimal sliding regimes arise, which are of great interest for the theory of optimal processes.

1°. Let Ω be the class of all real piecewise-continuous functions $u(t)$, $0 \leq t < \infty$. Denote $e^{iu(t)}$ by μ and write the equation

$$\dot{w} = -w\frac{\mu + w}{\mu - w}, \qquad w|_{t=0} = z. \tag{1}$$

Putting here $w = \omega(z,t)e^{-t}$, we have

$$\dot{\omega} = -\omega K(\omega, e^{t+iu}), \qquad K(z,\zeta) = 2z(\zeta - z)^{-1}. \tag{2}$$

K. Löwner showed ([3,4]) that, for any fixed $t > 0$, the integrals of equation (1) map the disk $|z|<1$ univalently and conformally onto the disk $|w|<1$, the functions $\omega(z,t)$ are functions of the class S, and for any $\varepsilon > 0$ and $f \in S$

there is a function $u(t)$ in Ω for which $|f(z)-\omega(z,\infty)|<\varepsilon$ in the disk $|z|\leq 1/2$. Therefore, to solve problem A it is enough to find it for functions $f(z)=\omega(z,\infty)$. By the method of interior variations ([4]) one can prove that this solution exists and, as a function of z, maps the disk $|z|<1$ onto a plane with piecewise-analytic cuts.

2°. Instead of $K(\omega, e^{t+iu})$, for brevity, we shall write $K(\omega)$. The transformations of this paragraph are valid for any function $K(\omega)$ regular at the point $\omega=0$. Comparing the coefficients of z^n on the right- and left-hand sides of the equality

$$\frac{\partial}{\partial t}G(\omega)=-G'(\omega)K(\omega),$$

we obtain

$$G\dot{y}=-GDK(F)y, \tag{3}$$

where G, D, F, and y denote matrices of sizes respectively $1\times(n+1)$, $(n+1)\times(n+1)$, $(n+1)\times(n+1)$, and $(n+1)\times 1$ with elem–

* Here and below, primes denote differentiation with respect to z, a dot denotes differentiation with respect to t, and by $\{A(z)\}_n$ we denote the coefficient of z^n in the Maclaurin expansion of any function $A(z)$ regular at $z=0$.

where $g_s=\{G(z)\}_s$, $d_{s,r}=0$ for $s\neq r$, $d_{s,s}=1$, $f_{s,r}=0$ for $s\neq r-1$, $f_{s,s+1}=1$, $y_s=\{\omega^s(z,t)\}_n$, $s,r=0,1,\dots,n$.* Taking all that has been said into account, problem A can be formulated in terms of the theory of optimal processes:

B. Find the minimum of the functional

$$J=\operatorname{Re} Gy(\infty) \tag{4}$$

over all $u\in\Omega$ on solutions of the system of equations

$$\dot{y}=-DK(F,e^{x+iu})y, \qquad \dot{x}=1,$$
$$x=y_0=\dots=y_{n-1}=y_n-1=0 \quad \text{for } t=0. \tag{5}$$

3°. L. S. Pontryagin's maximum principle for problem B is formulated as follows: if the function $u_0(t)$ delivers a minimum of the functional J, then for all $\tau\geq 0$ and real v

$$H(v,\tau)\leq 0, \qquad H(u_0(\tau),\tau)\equiv 0, \tag{6}$$

where

$$H(v,t)=\varphi(t)+\operatorname{Re}\left[\psi(t)DK(F,e^{x(t)+iv})y(t)\right], \tag{7}$$

and $\varphi(t), \psi(t)$ are integrals of the equations

$$\dot{\varphi} = -\partial H(u_0, t)/\partial x, \qquad \varphi(\infty) = 0,$$
$$\dot{\psi} = \psi D K(F, e^{x+iu_0}), \qquad \psi(\infty) = G. \tag{8}$$

Inequality (6) can be replaced by the condition

$$H(u_0(\tau), \tau) = \max_{-\infty<\vartheta<\infty} H(\vartheta, \tau). \tag{6'}$$

The maximum principle, as is known ([2]), for systems linear in the aggregate of the variables y, u, gives a necessary and sufficient condition for the minimum of J, while for nonlinear systems, in most cases, it gives a necessary condition "close" to sufficient (for the precise meaning of this closeness see ([5],[6])). System (5) may be called quasilinear—it is linear in y, but not linear in u.

4°. Let $\vartheta(t)$ be some differentiable function of class Ω, $v = e^{i\vartheta(t)}$, $\mu = e^{iu_0(t)}$, $\mu \neq v$ for no $t \geq 0$, and

$$h(t) = H(\vartheta(t), t) \equiv \varphi(t) - 2\,\mathrm{Re}\,[\psi(t) DF(e^{x(t)+i\vartheta(t)} - F)^{-1} y(t)]. \tag{9}$$

Let us compute $\dot{h}(t)$. Taking (5) and (8) into account, after simple algebraic transformations we find

$$\dot{h}(t) = \alpha H(u_0(t), t) + \beta H_\vartheta(u_0(t), t) + \gamma H(\vartheta(t), t) + \delta H_\vartheta(\vartheta(t), t), \tag{10}$$

where

$$-\alpha = \gamma = 4\mu v(\mu - v)^{-2}, \qquad \beta = i(v + \mu)(v - \mu)^{-1}, \qquad \delta = \beta + \dot{\vartheta}.$$

Analogously, for the function $h_1(t) = H_\vartheta(\vartheta(t), t)$ we establish that

$$\dot{h}_1(t) = \alpha_\vartheta H(u_0(t), t) + \beta_\vartheta H_\vartheta(u_0(t), t) + \gamma_\vartheta H(\vartheta(t), t) + (\gamma +$$
$$+ \delta_\vartheta) H_\vartheta(\vartheta(t), t) + \delta H_{\vartheta^2}(\vartheta(t), t). \tag{10_1}$$

Formally, (10_1) can be obtained from (10) by differentiating with respect to ϑ. In general, if $h_k(t) = H_{\vartheta^k}(\vartheta(t), t)$, $k = 1, 2, \ldots$, then

$$\dot{h}_k(t) = \alpha_{\vartheta^k} H(u_0(t), t) + \beta_{\vartheta^k} H_\vartheta(u_0(t), t) + \delta h_{k+1}(t) +$$
$$+ (\text{a linear combination of } h(t), h_1(t), \ldots, h_k(t)). \tag{10_k}$$

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* It is easy to see that the matrices F and D do not commute and that $K(F)D = DK(F) + FK'(F)$.

At the same time, according to (6), in the right-hand sides of equations (10)–(10_k) the first two terms vanish.

Let us now suppose that in system (2), on some interval (t', t''), an optimal "sliding" regime is possible, i.e., for $t' < t < t''$, together with $u_0(t)$, an absolute maximum of the function $H(\vartheta, t)$ is also provided by some $\vartheta(t) \neq u_0(t)$. It can be shown that the function $\vartheta(t)$, $t' < t < t''$, is then differentiable. Constructing from it the function (9), we shall have $h(t) \equiv h_1(t) \equiv 0$. Turning then to equations (10_k), $k = 1, 2, \ldots$, we conclude that

$$\delta \equiv -i\left(\frac{\dot{\nu}}{\nu} + \frac{\mu+\nu}{\mu-\nu}\right) \to 0, \tag{11}$$

i.e., $\nu = e^{i\vartheta(t)}$ turns out to be an integral of equation (1).

5°. Having obtained equation (11), it is natural to study the behavior of the function (9) for an arbitrary integral $\nu = e^{i\vartheta(t)}$ of equation (11). According to (10), (11), in this case

$$\dot{h}(t) = \gamma h(t),$$

and if by $\chi(t)$ we denote some integral of the equation

$$\dot{\chi} = -\frac{4\mu\nu}{(\mu-\nu)^2}\chi$$

then, evidently, it will turn out that

$$\chi(t)h(t) \equiv \text{const}\,. \tag{12}$$

The invariant (12), as G. S. Goodman indicated ([7]), is the equation of M. Schiffer for problem A. In turn, M. Schiffer' s equation is the principal result that can be obtained by solving problem A by the variational methods of the geometric theory of functions.

6°. Suppose that for $t = 0$ the functions $\varphi(t)$ and $\psi(t)$ are such that the function $\bar{H}(\vartheta, 0)$ attains the value zero at several points at once: $u_0(0), \vartheta_1^0, \ldots, \vartheta_k^0$. Construct the integrals $\nu_k(t)$ of equation (11) with initial conditions $\nu_k(0) = e^{i\vartheta_k}$. Then, according to (12), we shall have $\bar{H}(\vartheta_k(t), t) \equiv 0$ for all t, where $\vartheta_k(t)$ is real and is not equal to $u_0(t)$. Thus, for optimal sliding regimes to appear in system (5), it is sufficient that simple initial conditions for $\varphi(t)$ and $\psi(t)$ be fulfilled. And since in the theory of optimal processes sliding regimes appear only in rare, exceptional cases ([8]), system (5) is of great interest for the theory of optimal processes and deserves careful study.

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Received
17 I 1969

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Note: Figure translations are in progress. See original paper for figures.