EXPANSION IN NORMAL WAVES IN A SPHERICALLY LAYERED MEDIUM

MATHEMATICAL PHYSICS

1969

SovietRxiv

Abstract

Full Text

UDC 534.8

MATHEMATICAL PHYSICS

P. E. KRASNOUSHKIN

EXPANSION IN NORMAL WAVES IN A SPHERICALLY LAYERED MEDIUM

(Presented by Academician I. M. Vinogradov, 21 VIII 1968)

1. The field of radio waves in a layered medium, excited by currents $j(M) \times \exp(-i\omega t)$, can be expanded in normal waves running along or across the layers ([1]). If the medium possesses waveguide properties ([2], [3]), then the first of these expansions is expedient. It can be obtained in two ways: 1) by the integral transformation (2)–(11) ([1]), starting from the expansion in normal waves running across the layers, and 2) by direct expansion in the spectrum of the normal-wave operator, called below the direct method. Until now only the first method has been regarded as rigorous. In ([4], [5]) this method was used to obtain expansions for several particular cases of dissipative media. The direct method was first applied in ([2]) for conservative media, and in ([3]) for dissipative media, but without sufficient justification. The modern theory of non-self-adjoint operators, which has undergone intensive development after the work ([6]), makes it possible to derive a rigorous solution of the problem by the direct method and to investigate this solution for a broad class of media. Here we do this for a spherically layered isotropic medium characterized by a scalar complex function of dielectric permittivity $\varepsilon(r,\omega) = \varepsilon' + i\varepsilon''$ ($\varepsilon'' > 0$; r, θ and φ are spherical coordinates), under the condition that, beginning with some r_∞, the function $\varepsilon(r)$ tends monotonically to 1 as $r \to \infty$. Restricting ourselves for simplicity to the case $j_\varphi = 0$ and $\partial j/\partial\varphi = 0$, we introduce the scalar potential B, defined by the equation

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial B}{\partial\theta}\right)\varepsilon r - \frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\frac{\partial B}{\partial r}\right) + k_0^2 r^2 \varepsilon B = \frac{4\pi r^2}{c}\left[j_r - \varepsilon\frac{\partial}{\partial r}\left(\frac{r}{\varepsilon}\int j_\theta\, d\theta\right)\right]. \tag{1}$$

The amplitudes of the components of the excited field are expressed in terms of B.

2. **The case of an impermeable core of the medium.** Let the region $r \le a$ (for example, a spherical Earth of radius $r = a$) be filled with a medium with $|\varepsilon| = \infty$. Then the boundary-value problem for equation (1)

is considered in the region $a \leq r < \infty$ with boundary $r = a$, where one may impose the condition $E_\theta/H_\varphi = 0$. Requiring that B belong to the class of functions with integrable square, i.e. $B \in \mathcal{L}_2(a \leq r < \infty,\ 0 \leq \theta \leq \pi)$, we impose, instead of the radiation condition as $r \to \infty$, the artificial condition $\lim \varepsilon = 1 + i\varepsilon''_\infty$, where ε''_∞ is a small positive constant, which we shall let tend to 0 in the solution $B(r, \theta; \varepsilon''_\infty)$. The formulated problem can be written in the form

$$L_r B - L_\theta B = -F(\theta, r), \tag{2}$$

where L_r is the "transverse" operator generated by the differential expression

$$l_r X = -\varepsilon r^2 \frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\frac{\partial X}{\partial r}\right) - k_0^2 r^2 \varepsilon X$$

and by the condition

$$Y'/Y = 0 \quad \text{at } r = a, \tag{3}$$

acting in $\mathcal{L}_2(a, \infty)$ with the metric defined by the scalar product

$$(X, U) = \int_a^\infty X\overline{U}\, dr \tag{4}$$

L_r is a singular dissipative operator, hereafter called the operator of normal waves.

L_θ is a self-adjoint operator, for which expansions in eigenfunctions are well known. It is precisely for this reason that in (5) the construction of the expansion in normal waves propagating along the layers was begun with an expansion in waves propagating across the layers, i.e., in the spectrum of L_θ, subsequently transformed into the desired expansion. In solving (2) by a direct method one would have to start from the contour integral (2) (1), for which it is necessary to construct the Green' s functions of the resolvents $(L_r - \lambda\bar{E})^{-1}$ and $(L_\theta + \lambda\bar{E})^{-1}$. However, these constructions, as well as the evaluation of the integral itself (2) (1), are nontrivial. We shall avoid them by using results of work on the study of spectra of non-self-adjoint differential operators and expansions in their eigenfunctions. For this purpose, by the substitution $B = \sqrt{\varepsilon r}\, C(r, \theta)$ and $\zeta = \ln r$ we bring (2) to the form

$$L_\zeta C - L_\theta C = -F/\sqrt{\varepsilon r}, \qquad l_\zeta = -\partial^2/\partial\zeta^2 + W(\zeta), \tag{5}$$

where

$$W(\zeta) = -[k_0^2 \varepsilon r^2 - \sqrt{\varepsilon r}(1/\sqrt{\varepsilon r})_{\zeta\zeta}].$$

L_ζ is the operator of the Schrödinger equation with a complex-valued potential function $W(\zeta)$. The spectra of L_r and L_ζ coincide. Since $|W(\zeta)| \to \infty$ and $\operatorname{Im} W(\zeta) > 0$ as $\zeta \to \infty$, the spectrum of the operator L_ζ (L_r) is purely discrete and lies in the upper half-plane of λ, having no other limit points except $\lambda = \infty$. This important property follows from Theorem 2 of (7). We note that it cannot be derived from quantum-mechanical considerations concerning a potential well with impenetrable barriers, used for self-adjoint operators. In the present case there is no potential barrier as $\zeta \to \infty$, and the discreteness of the spectrum is due to the monotone increase of dissipation as $\zeta \to \infty$, i.e., of the imaginary part W ($\varepsilon''_\infty \neq 0$!). This spectrum is unstable with respect to small perturbations of L_ζ at the singular point $\zeta = \infty$: as $\varepsilon''_\infty \to 0$ it disappears, and the spectrum of L_ζ becomes continuous. It follows from (8) that the system of eigenfunctions and associated functions $\{X_j(r)\}$ of the operator L_r will be complete in $\mathcal{L}_2$. However, it is not orthogonal, and from its completeness there does not follow convergence, which, incidentally, is not necessary in the normal-wave expansion (1). With the aid of Mukminov' s criterion (9) one can show that in the present case $\{X_j\}$ forms a Riesz basis for the linear closure of the functions under consideration, orthogonalizable by a completely continuous operator. Therefore, if F belongs to the linear closure $\{X_j\}$, then the expansion

$$B = \sum_j \Phi_j(\theta) X_j(r) \tag{6}$$

converges in the norm of $\mathcal{L}_2$. To compute Φ_j let us consider the operator L_r^*, adjoint to L_r, defined by the differential expression

$$l_r^* = \frac{\partial}{\partial r}\left[\frac{1}{\bar{\varepsilon}}\frac{\partial}{\partial r}(\bar{\varepsilon} r^2 \cdot)\right] + k_0^2 \bar{\varepsilon} r^2. \tag{7}$$

and by the complex-conjugate boundary conditions at $r = a$ and $r \to \infty$. Its eigenvalues χ_j and eigenfunctions U_j will be

$$\chi_j = \bar{\lambda}_j, \qquad U_j = \bar{X}_j / \varepsilon r^2; \tag{8}$$

here a bar denotes complex-conjugate quantities. Between X_j and U_p there holds the biorthonormality condition

$$(X_j, U_p) = N_j \delta_{j,p}, \tag{9}$$

where $\delta_{j,p}$ is the Kronecker symbol, and the normalizing factor N_j is equal to

$$N_j = \int_a^\infty X_j \overline{U_j}\, dr = \int_a^\infty \left(\frac{X_j^2}{\varepsilon r^2}\right) dr. \tag{10}$$

Substituting (6) into (2), then multiplying scalarly according to (4) by U_p, after using (9), we obtain for $\Phi_j(\theta)$ the equations

$$l_\theta \Phi_j - \lambda_j \Phi_j = F_j(\theta), \qquad \text{where} \quad F_j = \frac{1}{N_j} \int_a^\infty \left(\frac{F}{\varepsilon r^2} \right) X_j \, dr. \tag{11}$$

They are solved under boundedness conditions for $\theta = 0$ and $\theta = \pi$ with the aid of the fundamental functions $P_{\nu_j}(\cos\theta)$; $P_{\nu_j}[\cos(\pi-\theta)]$ of equation (11), where P_{ν_j} are Legendre functions of complex order ν_j, determined through λ_j: $\lambda_j = -\nu_j(\nu_j+1)$. This is done in detail in ([3]). As a result, (6) takes the form

$$B = -\frac{\pi}{2} \sum_j \frac{X_j(r)}{\sin \nu_j \pi} \left\{ P_{\nu_j}[\cos(\pi-\theta)] \int_0^\theta F_j P_{\nu_j}[\cos\theta'] \sin\theta' \, d\theta' + \right.$$

$$\left. + P_{\nu_j}[\cos\theta] \int_\theta^\pi F_j P_{\nu_j}[\cos(\pi-\theta')] \sin\theta' \, d\theta' \right\}. \tag{12}$$

In the case of a point source, i.e., an electric vertical Hertz dipole with moment P, which is located at the point $(\theta = 0,\ r = b)$, from (12) we obtain, by passage to the limit, as was done in ([2]), the expression:

$$B = -\frac{\pi}{c} \frac{P}{b^2} \sum_j \frac{X_j(r) X_j(b)}{N_j \sin \nu_j \pi} P_{\nu_j}[\cos(\pi-\theta)]. \tag{13}$$

Introducing the asymptotic approximation for P_{ν_j} and expanding $(\sin\nu_j\pi)^{-1}$ into a geometric progression, the first term of which gives the normal waves reaching the receiving point (θ, r) only along the shortest path (echo waves are discarded), we become convinced that $\nu^* = \nu + 1/2$ is the angular wave number of the normal wave, while (12) and (13) are expansions in normal waves propagating along the layers. Since $F = \delta(\theta)\delta(r-b)$ does not enter into the envelope $\{X_j\}$, (13) converges in norm everywhere except the point $\theta = 0$, where convergence in the generalized sense takes place ([10]). The special case of (13) for $\varepsilon = 1$ $(a < r < \infty)$ (the so-called diffraction problem) was obtained in ([5]) by the transform method. It was repeated by Sommerfeld in ([11]), and then in ([12]) by the method of direct expansion in normal waves (called eigenwaves). In doing so, the fact that the operator L_r is not self-adjoint was omitted, and instead of condition (9) the orthogonality condition for a self-adjoint operator was used. To obtain a result coinciding with ([5]), the authors of ([11,12]) had to define the scalar product incorrectly. The correct solution was given in ([3]). An analogous error was made in ([12]) in considering a cylindrical layered medium.

Normal waves of large numbers j become purely attenuating. Waves of the first numbers may be weakly attenuating if the corresponding λ_j have negative $\operatorname{Re}\lambda$ and small $\operatorname{Im}\lambda$. This occurs in the waveguide case, when in a sufficiently

Fig. 1

Figure 1: Fig. 1

wide interval $\xi_1 < \xi < \xi_2$ the function $\mathrm{Re}(\lambda - W)$ becomes negative. The region (ξ_1, ξ_2) is a semitransparent potential barrier that impedes the loss of wave energy to radiation into infinity. A simple way of isolating the waveguide part of the discrete spectrum consists in straightening the layers of the medium $(a \to \infty)$ while preserving the profile $\varepsilon(r)$. In this case the normal-wave operator L_r, transformed to the Schrödinger form, takes the form L_z

$$l_z = -\partial^2/\partial z^2 + W_0; \qquad W_0 = \left[-k_0^2\varepsilon - \sqrt{\varepsilon}\,(1/\sqrt{\varepsilon})''_{zz}\right], \tag{14}$$

where z is the cylindrical coordinate directed along the radius r. Replacing the parameter λ by $\tilde{\lambda} = \lambda + k_0^2$, we obtain, instead of W_0, the potential function $\widetilde{W}_0 = W_0 - k_0^2$, which as $z \to \infty$ tends to the small quantity $k_0^2\varepsilon''_\infty i$. According to Theorem I of [7], the spectrum of such an operator consists of a continuous part occupying part of the real half-axis $\tilde{\lambda} > 0$, and of a certain bounded discrete set of eigenvalues $\{\tilde{\lambda}_j^0\}$ in the upper half-plane, which determine the **waveguide normal waves** (the linear wave numbers $\nu_j^0 = \sqrt{-\tilde{\lambda}_j^0}$ of these waves at $r = a$ are related to the conditional ν_j^0 by the relation $\gamma_j^0 = \nu_j^0/a$). If there are no multiple λ_j, then the waveguide numbers ν_j can be singled out from the total spectrum $\{\lambda_j\}$ by establishing a continuous correspondence between $\nu_j(a)$ and $\nu_j^0 = \nu_j(a \to \infty)$, varying a from the prescribed value to $a = \infty$. The waveguide spectrum will be finite if $\int_a^\infty \exp \mu z\, |\widetilde{W}_0|\, dz < \infty$. As the potential barrier is weakened in Watson' s waveguide problem [5], or when for $r > c$, i.e., in the ionosphere, $\varepsilon_i'' \to 0$, $\varepsilon_i' \to 1$, the waveguide spectrum merges with the diffraction spectrum (Fig. 1). This occurs in a nontrivial way, since in this case two spectra merge into one: the waveguide spectrum associated with the interval (a, c) (line B_1), and the diffraction spectrum associated with the interval (c, ∞) (line B_2); D is the line of the diffraction spectrum for $\varepsilon_i'' = 0$, $\varepsilon_i' = 1$.

Fig. 1

3. **The case of a permeable core of the medium.** If ε is finite everywhere, then one should consider the operator L_r on $(0, \infty)$. It has two singular points, $r = 0$ and $r = \infty$. To study its spectrum one may apply the splitting method [13]. For this, introducing the additional boundary condition $X(a) = X'(a)$, at the point a we decompose the resulting L'_ζ into the orthogonal sum of two operators $L'_\zeta = L_1 \oplus L_2$ in $\mathcal{L}_2(-\infty, \ln a)$ and $\mathcal{L}_2(\ln a, \infty)$. As $\zeta \to -\infty$, the function $W \to \mathrm{const}$. Hence, by Theorem I [7], we obtain that the spectrum of L_r has a continuous part filling the real half-plane for $\lambda > 0$ (the imaginary half-plane $\nu > 0$). Now the expansion in normal waves consists of a discrete sum, analogous to (13), and an integral over the continuous spectrum of normal waves.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
20 VIII 1968

CITED LITERATURE

1. P. E. Krasnushkin, DAN, 185, No. 5 (1969).
2. P. E. Krasnushkin, DAN, 56, No. 7, 687 (1947); *The Method of Normal Waves in Application*, Moscow State University, 1947.
3. P. E. Krasnushkin, N. A. Yablochkin, *Theory of Propagation of Superlong Waves*, Moscow, 1955; 2nd ed., Computing Center of the Academy of Sciences of the USSR, 1963.
4. A. Sommerfeld, Ann. Phys., 28, 665 (1909); 81, 1135 (1926).
5. G. N. Watson, Proc. Roy. Soc., A, 95a, 83 (1918); 95, 546 (1919).
6. M. V. Keldysh, DAN, 77, 11 (1951).
7. M. A. Naimark, DAN, 85, No. 1, 41 (1952).
8. M. V. Keldysh, V. B. Lidskii, Proc. of the IV All-Union Mathematical Congress, 1, 1963, p. 101.
9. B. R. Mukminov, DAN, 99, No. 4 (1954).
10. V. B. Lidskii, Trans. Moscow Math. Soc., 11, 3 (1962).
11. A. Sommerfeld, *Partial Differential Equations in Physics*, IL, 1950 (1st German ed. 1948).
12. G. T. Markov, A. F. Chaplin, *Excitation of Electromagnetic Waves*, 1967.
13. I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, 1963.
14. P. E. Krasnushkin, R. B. Baibulatov, DAN, 182, No. 2 (1968).

Note: Figure translations are in progress. See original paper for figures.