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# ON NONLINEAR MULTIDIMENSIONAL DIFFERENTIAL EQUATIONS

MATHEMATICS

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## Abstract

## Full Text

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MATHEMATICS

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# ON NONLINEAR MULTIDIMENSIONAL DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev on 11 IV 1969)

The paper gives sufficient conditions for the existence of an almost periodic solution of a nonlinear multidimensional differential equation with an almost periodic right-hand side. Preliminary study is made of conditions for complete integrability of such equations and of conditions for the applicability of the small-parameter method.

The following notation is adopted in the paper.  $E_x$  and  $E_y$  are real and complex finite-dimensional Banach spaces;  $L(E_x; E_y)$  is the space of linear operators acting from  $E_x$  into  $E_y$ ;  $L[E_x; E_y] = L(E_x; L(E_y; E_y))$ . The space of  $j$ -linear symmetric operators defined on  $E_y$  and with values in  $L(E_x; E_y)$  is denoted by  $S_j[E_x; E_y]$ ; if  $f \in S_j[E_x; E_y]$ , then  $\|f\|_* = \max\{\|fy^j\|/\|y\|^j\}$ . The symmetric and cosymmetric parts of a multilinear operator  $f$  are denoted by  $\vee f$  and  $\wedge f$ , respectively. If  $y(x)$  is a bounded function defined on  $E_x$  and with values in  $E_y$ , then  $\|y(\cdot)\| = \sup \|y(x)\|$ .

For information used in the paper on vector analytic functions of a vector argument and power series, see <sup>(1)</sup>, and on vector almost periodic functions of a vector argument, see <sup>(2,3)</sup>.

**1. Complete integrability.** Consider the nonlinear multidimensional differential equation

$$y' = f(x, y). \quad (1)$$

Suppose that

$$f(x, y) = \sum_{0 \leq j} f_j(x) y^j, \quad (2)$$

where  $f_j(x)$  is an operator almost periodic function defined on  $E_x$  and with values in  $S_j[E_x; E_y]$ ,

$$f_j(x) \sim \sum f_j^\lambda e^{i\lambda x}. \quad (3)$$

Here  $f_j^\lambda \in S_j[E_x; E_y]$ , and  $\lambda$  runs through the spectrum  $\sigma_j \subset E_x^*$  of the function  $f_j(x)$ .

If the operator function  $f(x, y)$  is continuously differentiable in both variables, then, according to the Frobenius theorem (see, for example, <sup>(4)</sup>), complete integrability holds if and only if

$$\bigwedge_{hk} \left\{ \frac{\partial f(x, y)}{\partial x} hk + \frac{\partial f(x, y)}{\partial y} [f(x, y)h]k \right\} = 0 \quad (\forall h, k \in E_x). \quad (4)$$

Substituting expression (2) into this condition and taking (3) into account, we arrive at the system of relations

$$\bigwedge_{\lambda hk} \left\{ i\lambda h f_j^\lambda y^j h + \sum_{\substack{a+b=j+1 \\ \mu+\nu=\lambda}} a f_\mu^a y^{a-1} (f_b^\nu y^b h) k \right\} = 0 \quad (5)$$

$$[\forall h, k \in E_x, \quad \forall y \in E_y; \quad j = 0, 1, 2, \dots; \quad \lambda \in \sigma + \sigma),$$

where

$$\sigma = \bigcup_{0 \leq j} \sigma_j.$$

**Theorem 1.** Let  $f_j(x)$  be an almost periodic operator function defined on  $E_x$  and with values in  $S_j[E_x; E_y]$  ( $j = 0, 1, 2, \dots$ ), and suppose that, for some  $r$ , for every  $\delta$ ,  $0 < \delta < 1$ , the series (2) converges uniformly for  $x \in E_x$ ,  $\|y\| \leq (1 - \delta)r$ .

Then the complete solvability of equation (1) holds if and only if condition (5) is satisfied.

We emphasize that under the hypotheses of Theorem 1 the operator function  $f(x, y)$  need not be differentiable with respect to  $x$ .

**Theorem 2.** The operator function  $f(x, y)$  is representable in the form (2) and satisfies the hypotheses of Theorem 1 if and only if the following conditions are satisfied: in the domain  $x \in E_x$ ,  $\|y\| < r$ , the function  $f(x, y)$  is differentiable with respect to  $y$  and almost periodic with respect to  $x$ ; for every  $\delta$ ,  $0 < \delta < 1$ , one can specify a constant  $M_\delta$  such that  $\|f(x, y)\| \leq M_\delta$  for  $x \in E_x$ ,  $\|y\| \leq (1 - \delta)r$ .

**2. The small-parameter method.** Consider the nonlinear multidimensional differential equation containing a small complex parameter  $\varepsilon$ ,

$$y'h = Ah y + \varphi(x)h + \varepsilon f(x, y, \varepsilon)h \quad (\forall h \in E_x), \quad (6)$$

where  $A \in L[E_x; E_y]$ ,  $\varphi(x)$  is an almost periodic operator function with values in  $L(E_x; E_y)$ ;

$$f(x, y, \varepsilon) = \sum_{0 \leq p, q} f_{pq}(x) y^p \varepsilon^q \quad (7)$$

and  $f_{pq}(x)$  is an almost periodic operator function with values in  $S_p[E_x; E_y]$  ( $p, q = 0, 1, 2, \dots$ ).

Suppose that  $\varphi(x)$  and  $f(x, y, \varepsilon)$  are differentiable in the domain  $x \in E_x$ ,  $\|y\| < r$ ,  $|\varepsilon| < s$ . From condition (4) it is not difficult to obtain that the complete solvability of equation (6) for all  $\varepsilon$ ,  $|\varepsilon| < s$ , holds if and only if the following conditions are satisfied:

$$\bigwedge_{hk} AhAk = 0, \quad (8)$$

$$\bigwedge_{hk} \{\varphi'(x)hk + Ak\varphi(x)h\} = 0, \quad (9)$$

$$\bigwedge_{hk} D_{pq}(x; y_1, \dots, y_p)hk = 0. \quad (10)$$

Here  $h, k \in E_x$  are arbitrary,  $p, q = 0, 1, 2, \dots$ , and

$$\begin{aligned} D_{pq}(x; y_1, \dots, y_p)hk &= f'_{pq}(x)hy_1 \dots y_{pk} + Akf_{pq}(x)y_1 \dots y_{ph} \\ &+ \sum_{1 \leq j \leq p} f_{pq}(x)y_1 \dots (Ah y_j) \dots y_{pk} + (p+1)f_{p+1, q}(x)y_1 \dots y_p(\varphi(x)h)k \\ &+ \bigvee_{y_1 \dots y_p} \sum_{\substack{t+a=p-1 \\ s+b=q-1}} \sum_{1 \leq j \leq t} f_{ts}(x)y_1 \dots (f_{ab}y_j \dots y_{j+a}h) \dots y_{pk}. \end{aligned} \quad (11)$$

We shall seek a solution  $y(x) = y(x; \varepsilon)$  of equation (6) in the form

$$y(x) = y_0(x) + \varepsilon y_1(x) + \dots + \varepsilon^j y_j(x) + \dots \quad (12)$$

To determine the coefficients of this expansion we obtain the following system of linear nonhomogeneous equations:

$$y'_0(x)h = Ah y_0(x) + \varphi(x)h,$$

$$y_1'(x)h = Ah y_1(x) + \sum_{0 \leq p} f_{p0}(x)[y_0(x)]^p h, \quad (13)$$

$$y_j'(x)h = Ah y_j(x) + \sum_{j_1 + \dots + j_p + q + 1 = j} f_{pq}(x) y_{j_1}(x) \dots y_{j_p}(x) h$$

$$(j = 2, 3, \dots).$$

We shall say that system (13) is **completely solvable** if for any  $\xi \in E_x$  and  $y_0, y_1, \dots \in E_y$  there exists a solution  $y_0(x), y_1(x), \dots$  of this system for which  $y_j(\xi) = y_j$  ( $j = 0, 1, 2, \dots$ ).

As before, let us write the condition for complete solvability of the  $j$ -th equation of system (13) and in it replace  $y_0(x), \dots, y_{j-1}(x)$  by the vectors  $y_0, \dots, y_{j-1}$ . Under the sign  $\wedge$  we shall obtain a certain expression  $\mathcal{D}_j(x; y_0, \dots, y_{j-1})hk$ .

**Theorem 3.** Let  $f_{pq}(x)$  be an almost periodic operator function with values in  $S_p[E_x; E_y]$  ( $p, q = 0, 1, 2, \dots$ ), and suppose that for some  $r, s > 0$ , for every  $\delta, 0 < \delta < 1$ , the series (7) converges uniformly for  $x \in E_x, \|y\| \leq (1 - \delta)r, |\varepsilon| \leq (1 - \delta)s$ .

Then the system of equations (13) is completely solvable if and only if equation (6) is completely solvable.

The proof of this theorem in the smooth case is based on the formula

$$\wedge_{hk} \mathcal{D}_j(x; y_0, \dots, y_{j-1})hk = \sum p! \wedge_{hk} D_{pq}(x; y_{j_1}, \dots, y_{j_p})hk, \quad (14)$$

where the summation is over all possible sets of indices  $j_1, \dots, j_p$  with

$$j_1 + \dots + j_p + q = j - 1.$$

**Theorem 4.** The operator function  $f(x, y, \varepsilon)$  is representable in the form (7) and satisfies the conditions of Theorem 3 if and only if in the domain  $x \in E_x, \|y\| < r, |\varepsilon| < s$  the function  $f(x, y, \varepsilon)$  is differentiable with respect to  $y, \varepsilon$  and almost periodic with respect to  $x$ , and for every  $\delta, 0 < \delta < 1$ , one can indicate such a constant  $M_\delta$  that

$$\|f(x, y, \varepsilon)\| \leq M_\delta$$

for  $x \in E_x, \|y\| \leq (1 - \delta)r, |\varepsilon| \leq (1 - \delta)s$ .

### 3. Existence of an almost periodic solution

Consider the linear multidimensional differential equation

$$y'h = Ah y + \varphi(x)h \quad (\forall h \in E_x), \quad (15)$$

where  $A \in L[E_x; E_y]$  and  $\varphi(x)$  is an almost periodic operator function with values in  $L(E_x; E_y)$ . Denote by  $\Phi_\alpha(A)$  the totality of all almost periodic functions  $\varphi(x)$  ensuring complete solvability of equation (15) and satisfying the condition

$$\rho(\sigma(A), i\sigma\{\varphi(\cdot)\}) \geq \alpha > 0$$

( $\sigma(A)$  is the spectrum of the operator  $A$  <sup>5</sup>;  $\sigma\{\varphi(\cdot)\}$  is the spectrum of the function  $\varphi(x)$ ).

**Theorem 5.** For any  $\varphi(x) \in \Phi_\alpha(A)$ , equation (15) has a unique almost periodic solution  $y(x)$ , whose spectrum coincides with the spectrum of  $\varphi(x)$ . There exists a constant  $c = c(A, \alpha) > 0$  such that the estimate

$$\|y(\cdot)\| \leq c\|\varphi(\cdot)\|$$

is valid.

Let  $\sigma^+$  be the totality of all  $\lambda \in E_x^*$  representable in the form

$$\lambda = a_1\lambda^1 + \dots + a_n\lambda^n,$$

where  $a_1, \dots, a_n \geq 0$  are integers and  $\lambda^1, \dots, \lambda^n \in \sigma$  (see § 1). Suppose that

$$\rho(\sigma(A), i\sigma^+) \geq \alpha > 0.$$

Then each of the equations of the system (13), by Theorem 5, has a unique almost periodic solution  $y_j(x)$ , whose spectrum lies in  $\sigma^+$ .

To prove the existence of an almost periodic solution of equation (6), we shall apply the method of majorants. Alongside equation (6), consider the scalar equation

$$\zeta = c \left( \|\varphi(\cdot)\| + \varepsilon \sum_{0 \leq p, q} \|f_{pq}(\cdot)\|_* \zeta^p \varepsilon^q \right), \quad (16)$$

the right-hand side of which we denote by  $F(\zeta, \varepsilon)$  (see (6)).

**Theorem 6.** Suppose the conditions of Theorem 3 are fulfilled and equation (6) is completely solvable for  $|\varepsilon| < s$ . Let

$$\rho(\sigma(A), i\sigma^+) \geq \alpha > 0$$

and let  $c = c(A, \alpha)$  be the constant from Theorem 5.

Then: 1) if, for some  $\varepsilon^* > 0$ , the power series  $F(\xi, \varepsilon^*)$  has a nonzero radius of convergence and the equation  $\xi = F(\xi, \varepsilon^*)$  has a nonnegative root  $\xi^*$ , then the series (12) converges uniformly for  $|\varepsilon| \leq \varepsilon^*$  and represents an almost periodic solution whose spectrum lies in  $\sigma^+$ ;

- 2) if  $\xi^*$  is a simple root and  $(\xi^*, \varepsilon^*)$  is an interior point of convergence of the series  $F(\xi, \varepsilon)$ , then what has been stated above is valid for  $|\varepsilon| < \varepsilon^* + \Delta$ , where  $\Delta$  is a sufficiently small positive number.
4. The theory set forth above carries over without particular difficulty to the case when  $E_x$  and  $E_y$  are infinite-dimensional Banach spaces. The authors suppose that, under a corresponding generalization of the concept of solution, the theorems obtained will also remain valid if almost periodicity is understood in the sense of Stepanov, Weyl, or Besicovitch.

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