A GENERALIZED CONTINUOUS INTERVAL AND ITS APPLICATION IN HOMOTOPY THEORY

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 513.83

MATHEMATICS

E. D. KHALIMSKII

A GENERALIZED CONTINUOUS INTERVAL AND ITS APPLICATION IN HOMOTOPY THEORY

(Presented by Academician P. S. Aleksandrov on 5 VII 1968)

§ 1. By an interval (a generalized continuous interval) $\bar{A}$ we shall mean a connected space A satisfying the following conditions:

1. In A there are at most two points a_1, a_2 (called the endpoints of the interval) such that each of the spaces $A - \{a_i\}$, where $i = 1, 2$, is connected.
2. For every point a belonging to A and not coinciding with either of the points indicated in item 1, the space $A - \{a\}$ consists of exactly two connected components.

An interval $\bar{A}$ with two endpoints a_1 and a_2 will be denoted by $[\overline{a_1, a_2}]$.

The following theorem is easily proved:

Theorem 1. *Let each of the points* a_1, a_2 *belong to the interval* $\bar{A}$. *If* $a_1 \neq a_2$, *then the connected component of the space* $\bar{A} - \{a_2\}$ *not containing* a_1 *is contained in the same connected component of the space* $\bar{A} - \{a_1\}$ *as* a_2.

Take on the interval $\bar{A}$ a point a_1. Denote the connected components of the space $\bar{A} - \{a_1\}$ by U_{a_1}, U'_{a_1}; for an arbitrary point $a \in U_{a_1}$, denote by U_a that connected component of the space $A - \{a\}$ to which the point a_1 belongs, and denote the second by U'_a; if $a \in U'_{a_1}$, then conversely. For every point $a \in \bar{A}$ we shall assume that $b < a < c$, where b is an arbitrary point from U_a, and c from U'_a. It is easy to see that this gives an order relation on $\bar{A}$. If U_{a_1} is replaced by U'_{a_1} and conversely, then the order introduced is replaced by the opposite one. These two order relations will be called those induced by the connectedness of the interval. It follows from Theorem 1 that if, instead of the point a_1, one takes a point a_2 different from it, then each of the order relations obtained thereby coincides with one of the order relations obtained earlier. An oriented interval $\vec{A}$ is an interval $\bar{A}$ with one of the order relations thus introduced on it. In what follows we shall consider only oriented intervals and call them simply intervals. A subspace of an interval will be called a subinterval if it is itself an interval. The following propositions are easily proved:

Proposition 1. *A connected space A is an interval $\bar{A}$ if and only if the set of its points can be ordered so that connected in it will be precisely those and only those sets of points which, together with any two points a and b such that $a < b$, contain all points of A lying between them.*

Proposition 2. *A connected subspace of an interval is a subinterval.*

The topology in an ordered* set A is called interval, if open in it are only all possible unions of sets, each of which has one of the following forms:

$$] \leftarrow, a[, \ ]a, \rightarrow [, \ ]a_1, a_2[, \ A, \emptyset. \tag{1}$$

---

* In this paper only linearly ordered sets are considered.

Theorem 2.1. a) *If, for every cut (A, B) (^{2}) in an ordered set C, either A has a last element, or B has a first one, and one of these possibilities excludes the other, then one can introduce on C a T_1-topology under which C becomes a segment $\overline{C}$, one of whose orders coincides with the given order of its points in the ordered set.* b) *In the contrary case one cannot introduce on C a T_1-topology satisfying the condition described in item 1a).*

2. *If a T_1-topology satisfying the condition of item 1a) can be introduced, then the interval topology is the weakest of all T_1-topologies satisfying this condition.*
3. *In order that an ordered set C with some T_1-topology finer than the interval topology satisfy the condition of item 1a), it is necessary and sufficient that the sets U open in C in this topology satisfy the following two conditions: a) in U there is no point a_0 for which the intersection with U of at least one of the intervals of the form $]a_1, a_0[$ or $]a_0, a_2[$ would be empty (where $a_1 < a_0 < a_2$); b) the complement of a nonempty intersection of an arbitrary open set with an interval, relative to the whole interval, is not open.*

By a complete connected oriented union of the ordered set $\Lambda = \{\lambda\}$ of pairwise disjoint segments with T_1-topology

$$\overline{\vec{U}\vec{A}_\lambda}$$

we shall mean a $\vec{B}$ with a T_1-topology such that:

1. For every $\lambda \in \Lambda$ there exists a subsegment $\vec{B}_\lambda$ of the segment $\vec{B}$ and a homeomorphism
$$f_\lambda : \vec{A}_\lambda \rightarrow \vec{B}_\lambda,$$
where, if $\lambda_1 < \lambda_2$ in Λ and $a_1 \in \vec{A}_{\lambda_1}$, $a_2 \in \vec{A}_{\lambda_2}$, then
$$f_{\lambda_1}(a_1) < f_{\lambda_2}(a_2).$$

2. If $a_1 \in \vec{A}_\lambda$, $a_2 \in \vec{A}_\lambda$ and $a_1 < a_2$, then
$$f_\lambda(a_1) < f_\lambda(a_2).$$
3. For every λ there exists
$$[b_\lambda, b'_\lambda] \subset \vec{B}$$
such that one and only one of the following relations holds:

   a) $\vec{B}_\lambda = [b_\lambda, b'_\lambda]$; b) $\vec{B}_\lambda =]b_\lambda, b'_\lambda[$; c) $\vec{B}_\lambda = [b_\lambda, b'_\lambda[$; d) $\vec{B}_\lambda =]b_\lambda, b'_\lambda]$.
4. For every $b \in]b_\lambda, b'_\lambda[$ the relation
$$b_\lambda < b < b'_\lambda$$
holds.
5. If λ_2 immediately follows λ_1, then
$$b_{\lambda_2} = b'_{\lambda_1}.$$
6. If
$$\emptyset \neq]b', b''[\subset B,$$
then there exists $\lambda \in \Lambda$ such that
$$]b', b''[\cap]b_\lambda, b'_\lambda[\neq \emptyset.$$
7. The topology in $\vec{B}$ is such that: a) sets of each of the forms $] \to, b[$, $]b, \to [$, $]b_1, b_2[$, where each of the elements b_1, b, b_2 belongs to $\vec{B}$; sets open in any $\vec{B}_\lambda$ and distinct from $\vec{B}_\lambda$, and if $\vec{B}_\lambda =]b_\lambda, b'_\lambda[$, then also it itself, where $\lambda \in \Lambda$, as well as all possible unions of the sets listed above, are open sets in $\vec{B}$.

Theorem 3. *For every ordered set $\Lambda = \{\lambda\}$ of segments $\vec{A}_\lambda$ with T_1-topology, one can construct a segment $\vec{B}$ that is a complete connected oriented union of these segments.*

A generalized path in a space X is a mapping
$$f : \vec{A} \to X,$$
where $\vec{A}$ is an arbitrary segment. A space will be called generalized linearly connected if for any two points x_1, x_2 of X there exists a generalized path
$$f : [a_1, a_2] \to X$$
satisfying the conditions
$$f(a_1) = x_1, \qquad f(a_2) = x_2.$$

The generalized cube $e_n^{\vec{A}}$ is the Tikhonov product

$$\vec{A} \times \dots \times \vec{A},$$

where $\vec{A} = \overline{[a,b]}$ is a segment with T_1-topology.

§ 2. We shall call the boundary of the cube $e_n^{\vec{A}}$, and denote it by $\dot{e}_n^{\vec{A}}$, the set of points of $e_n^{\vec{A}}$ one of whose coordinates coincides with one of the endpoints of the segment $\vec{A}$. We shall consider T_1-segments and generalized linearly connected T_1-spaces and call them, respectively, segments and spaces.

We shall divide* the class of mappings of the form $f : \dot{e}_n \times \overrightarrow{[a_1, a_2]} \to X$ such that $f((a, \dots, a) \times \overrightarrow{[a_1 a_2]}) = x_0$ and $f(\dot{e}_n \times a_i) = x_0$ (where $x_0 \in X$, $i = 1, 2$; $n = 1, 2, 3, \dots$) into equivalence classes: two mappings $f_i : \dot{e}_n^i \times \vec{A}_i$, $(\dot{e}_n^i \times \vec{A}_i) \cup ([a^i, \dots, a^i] \times \vec{A}_i) \to X, x_0$ $(i = 1, 2)$ will be considered equivalent if one of the following three conditions is satisfied.

I. There exist an n-cube e_n, intervals $\overrightarrow{[a_1, a_2]}$ and $\vec{A}$, nondecreasing mappings $g_i : \vec{A}_i \xrightarrow{\text{onto}} \vec{A}*$, and a mapping $F : \dot{e}_n \times \vec{A} \times [a_1, a_2] \to X**$ such that the relations hold

$$F(\dot{e}_n \times g_i(x) \times a_i) = f_i(\dot{e}_n^i \times x), \qquad \text{where } x \in \vec{A}_i,\ i = 1, 2.$$

II. There exist an n-cube e_n, intervals $\overrightarrow{[a_1, a_2]}$ and $\vec{A}$, nondecreasing mappings $g_i : \vec{A} \xrightarrow{\text{onto}} \vec{A}_i$, and a mapping $F : \dot{e}_n \times \vec{A} \times [a_1, a_2] \to X$ such that the relations hold

$$F(\dot{e}_n \times x \times a_i) = f_i(\dot{e}_n^i \times g_i(x)), \qquad \text{where } x \in \vec{A},\ i = 1, 2.$$

III. There exists a partition of each of the intervals $\vec{A}_1, \vec{A}_2$ into the same number of parts $\vec{A}_i = \bigcup_{j=1}^k \vec{A}_i^j$ such that $f_i^j : \dot{e}_n^i \times \vec{A}_i^j, \dot{e}_n^i \times \vec{A}_i^j \to X, X_0$, where each pair of mappings f_1^j, f_2^j, induced respectively by the mappings f_1, f_2, consists of mappings equivalent to one another in the sense of one of the items I, II.

This equivalence relation divides the class of all mappings under consideration into pairwise disjoint equivalence classes. The mapping $f : \dot{e}_n \times \vec{A} \to X$ will be called the product of the mappings $f_1 : \dot{e}_n \times \vec{A}_1 \to X$ and $f_2 : \dot{e}_n \times \vec{A}_2 \to X$, and we shall denote it by $f = f_1 * f_2$, if $\vec{A} = \vec{A}_1 \cup \vec{A}_2$ and $f(\dot{e}_n \times a_i) = f_i(\dot{e}_n \times a_i)$, where $a_i \in \vec{A}_i \subset \vec{A}$, $i = 1, 2$.

From condition III it follows that one can introduce the product of equivalence classes $\{f_1\} * \{f_2\} = \{f_1 * f_2\}$. For each mapping $f : \dot{e}_n \times \vec{A} \to X$ we introduce, on the set whose elements are sets of the form $f(e_n \times c)$, where the point $c \in \vec{A}$, an order relation as follows: for any points $c_1 \in \vec{A}$ and $c_2 \in \vec{A}$ such that $c_1 < c_2$ and $f(e_n \times c_2) \neq f(e_n \times c_1)$, set $f(e_n \times c_1) < f(e_n \times c_2)$; the set whose elements are sets of the form $f(e_n \times c)$ (where $f : \dot{e}_n \times \vec{A} \to X,\ c \in \vec{A}$) with the order relation thus introduced is an ordered set. Two mappings $f, \bar{f} : e_n \times \vec{A}, e_n \times \vec{A} \to X, x_0$ will be called mutually opposite if for each set of the form $f(e_n \times a_1)$ there exists a point $a_2 \in \vec{A}$ such that $\bar{f}(e_n \times a_2) = f(e_n \times a_1)$, and conversely, for each set of the form $\bar{f}(e_n \times a_2)$ there exists a point $a_1 \in \vec{A}$ such that $\bar{f}(e_n \times a_2) = f(e_n \times a_1)$, moreover, if $f(e_n \times a_1) < f(e_n \times a'_1)$ and $\bar{f}(e_n \times a_2) = f(e_n \times a_1)$, $\bar{f}(e_n \times a'_2) = f(e_n \times a'_1)$, then $\bar{f}(e_n \times a_2) > \bar{f}(e_n \times a'_2)$, and conversely, if $\bar{f}(e_n \times a_2) > \bar{f}(e_n \times a'_2)$, then $f(e_n \times a_1) < f(e_n \times a'_1)$.

Obviously, if two mappings are equivalent, then the mappings opposite to them are equivalent as well. The equivalence classes $\{f\}, \{\bar{f}\}$ will be called mutually opposite. Mappings $e : \dot{e}_n \times \vec{A}, e_n \times \vec{A} \to X, x_0$ will be called degenerate if $e(\dot{e}_n \times \vec{A}) = x_0$.

* The mappings g_i are, generally speaking, not continuous.

** The mapping F satisfies the relation $F((a, \dots, a) \times \vec{A} \times \overrightarrow{[a_1, a_2]}) = x_0$.

Proposition 3. The classes of mutually equivalent mappings, none of which can be represented as a product of an infinite number of equivalence classes distinct from $\{e\}$, form a group with respect to the product

$$\{f_i\} * \{f_j\} = \{f_1 * f_2\};$$

the class of the degenerate mapping e serves as its identity element, and the inverse element to $\{f\}$ is $\{\bar{f}\}$*.

In an analogous way one can construct a theory of generalized singular cubical homology and of generalized relative homotopy groups.

Moscow State Pedagogical Institute
named after V. I. Lenin

Received
4 VII 1968

REFERENCES

1. N. Bourbaki, *Theory of Sets*, Moscow, 1958.
2. P. S. Aleksandrov, *Introduction to the General Theory of Sets and Functions*, Moscow, 1948.

---

* A class M of spaces is considered such that for every $X \in M$ there exists a cardinality $a(X)$ such that, for every f for which $\{f\}$ is not representable as a product of an infinite number of $\{f_i\} \neq \{e\}$, there exists

$$f_1 : e^{n_1} \times \bar{A}_1 \to X$$

equivalent to f, such that

$$\bar{e}_{n_1} \leq a(X), \qquad \bar{A}_1 \leq a(X).$$

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.