GROWTH OF MEROMORPHIC FUNCTIONS OF FINITE LOWER ORDER

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Abstract

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MATHEMATICS

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GROWTH OF MEROMORPHIC FUNCTIONS OF FINITE LOWER ORDER

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§ 1. Let $f(z)$ be a function meromorphic for $z \neq \infty$, of finite lower order λ and order ρ. R. Nevanlinna (^{1,2}) introduced the concept of the defect of $f(z)$ at the point a

$$\delta(a, f) = \lim_{r\to\infty} \frac{m(r, a, f)}{T(r, f)},$$

where $T(r, f)$ is the Nevanlinna characteristic of $f(z)$, and $m(r, a, f)$ is the mean proximity of $f(z)$ to a on the circle $|z| = r$ (see (^{2}), p. 169).

Obviously, the quantity $\delta(a, f)$ may be regarded as the magnitude of the mean deviation of $f(z)$ from the number a. Put, further,

$$M(r, a, f) = \max_{|z|=r} \frac{1}{|f(z) - a|} \quad \text{for } a \neq \infty,$$

$$M(r, \infty, f) = \max_{|z|=r} |f(z)|,$$

$$\beta(a, f) = \lim_{r\to\infty} \frac{\ln^+ M(r, a, f)}{T(r, f)}.$$

$\beta(a, f)$ characterizes the magnitude of the minimal deviation of $f(z)$ from the number a; we shall call $\beta(a, f)$ the magnitude of deviation of $f(z)$ from the number a.

The main results of the paper are:

Theorem 1. If a meromorphic function $f(z)$ is of finite lower order λ, then for an arbitrary complex number a

$$\beta(a,f) \leq \begin{cases} \pi\lambda/\sin\pi\lambda, & \text{for } 0 < \lambda < 0.5, \\ \pi\lambda, & \text{for } \lambda \geq 0.5. \end{cases} \tag{1,1}$$

$$\tag{1,2}$$

Theorem 2 (3). For an arbitrary fixed complex number a and for any fixed numbers λ and ρ such that

$$0.5 \leq \lambda \leq \rho \leq \infty,$$

there exists a meromorphic function $h_{\lambda,\rho,a}(z)$ of lower order λ and order ρ, for which

$$\beta(a, h_{\lambda,\rho,a}) = \pi\lambda.$$

Thus, estimate (1,2) is sharp. Estimate (1,1) for entire functions with λ replaced by ρ was proved by G. Valiron (4). In the general case, as noted by the author (5), estimate (1,1) follows from results of A. A. Gol' dberg and I. V. Ostrovskii (6,7). In 1932 Paley (8) proposed the hypothesis: if $g(z)$ is an entire function of order $\rho \geq 0.5$, then

$$\beta(\infty, g) \leq \pi\rho. \tag{1,3}$$

The validity of Paley' s hypothesis (i.e., the validity of estimate (1,3)) was recently proved by N. V. Govorov, as he reported at the function theory seminar of Kharkov University.

We shall henceforth denote by the letters K with subscripts positive absolute constants, and by the letters C with subscripts positive constants depending only on the function under consideration.

Let

$$\Omega(f) = \{a : \beta(a,f) > 0\}, \qquad \Delta(f) = \{a : \delta(a,f) > 0\}.$$

Obviously, $\Delta(f) \subseteq \Omega(f)$.

The following theorems characterize the set $\Omega(f)$ and the quantities $\beta(a,f)$.

Theorem 3. If $f(z)$ is a meromorphic function of finite lower order λ, then:

a) the set $\Omega(f)$ is at most countable;

b) for $\lambda < 0.5$,

$$\sum_{(a)} \beta^{\alpha}(a, f) \leq K_1 \frac{\lambda^{2\alpha}}{(2\alpha - 1)^{1/2}}, \tag{1,4}$$

if the number of points a for which $\beta(a, f) > 0$ is at least two;

c) for $\lambda \geq 0.5$,

$$\sum_{(a)} \beta^{\alpha}(a, f) \leq K_2 \frac{\lambda}{(2\alpha - 1)^{1/2}}, \tag{1,5}$$

where $1/2 < \alpha \leq 1$.

Theorem 4 (see (^{11}), p. 153). There exist meromorphic functions of finite lower order for which

$$\sum_{(a)} \beta^{\alpha}(a, f) = \infty$$

for every $\alpha < 0.5$.

Thus, only the question remains open as to whether the series

$$\sum_{(a)} \beta^{1/2}(a, f)$$

converges or may diverge for meromorphic functions of finite lower order.

In this direction the following is of interest.

Theorem 5. For a meromorphic function $f(z)$ of finite lower order λ, for every ε, $0 < \varepsilon \leq 1$, the estimate

$$\sum_{(a)} \frac{\beta^{1/2}(a, f)}{\ln^{1/2+\varepsilon}[e\pi(\lambda + 1)/\beta(a, f)]} \leq \frac{K_3}{\varepsilon}(\lambda + 1)^{3/2}. \tag{1,6}$$

Theorems 3, 4, and 5 may be regarded as analogues of the corresponding theorems for the deficiencies of meromorphic functions, obtained in the works ($^{1,2,9-12}$).

§ 2. We describe the method by means of which the main results of this paper are obtained (see also (^{5,9,10,12})). Let

$$D_{\alpha,R} = \{z : 0 < |z| < R,\ |\arg z| < \alpha\},$$

where $0 < \alpha < \pi$, $R > 1$.

Lemma 1. Let $g(z)$ be an analytic function in $\overline{D}_{\alpha,R}$, and let, for $z \in \overline{D}_{\alpha,R}$, $g(z) \neq 0, \infty$. If $\ln g(0) = 0$, then for $0 < s < R$

$$\ln|g(s)| = \frac{s^{2\varkappa}}{2\alpha}\int_0^R \frac{\arg g(te^{i\alpha}) - \arg g(te^{-i\alpha})}{t}\,\frac{dt}{s^{2\varkappa}+t^{2\varkappa}} -$$

$$-\frac{s^{2\varkappa}}{2\alpha}\int_0^R \frac{\arg g(te^{i\alpha}) - \arg g(te^{-i\alpha})}{t}\,\frac{t^{2\varkappa}}{t^{2\varkappa}s^{2\varkappa}+R^{4\varkappa}}\,dt -$$

$$-\frac{R^{2\varkappa}s^{2\varkappa}}{\alpha}\int_{-\alpha}^{\alpha} \arg g(Re^{i\theta})\,\frac{\sin 2\varkappa\theta\,d\theta}{4R^{2\varkappa}s^{2\varkappa}\sin^2\varkappa\theta + (R^{2\varkappa}-s^{2\varkappa})^2}, \qquad (2,1)$$

where $x = \pi/2\alpha$,

$$\frac{1}{2\pi}\int_{-\alpha}^{\alpha} \ln|g(se^{i\varphi})|\,d\varphi = \frac{1}{2\pi}\int_0^s \frac{\arg g(te^{i\alpha}) - \arg g(te^{-i\alpha})}{t}\,dt. \qquad (2,2)$$

Indeed, relation (2,1) is obtained in solving the Dirichlet problem for $\operatorname{Im}\ln g(z) = \arg g(z)$ in the domain $D_{\alpha,R}$ (see (^{9}), p. 1160, (^{10})), while formula (2,2) follows easily from Cauchy' s theorem for the function $\{\ln g(z)\}z^{-1}$, analytic in the domain $D_{\alpha,R}$.

For fixed r $(1 < r < R)$, put, for a meromorphic function $f(z)$ $(f(0) = 1)$ with $z \neq \infty$,

$$|f(re^{i\theta(r)})| = M(r,\infty,f), \qquad F_r(z) = f(e^{i\theta(r)}z) \quad (z = se^{i\varphi})$$

and choose a $(0 < a < \pi)$ and $R > 1$ so that the boundary of $D_{\alpha,R}$ is free of the zeros a_k and poles b_k of the meromorphic function $F_r(z)$.

Let $(x = \pi/2\alpha)$

$$\Phi_{2R}(z,F_r) = \prod_{a_k\in D_{\alpha,2R}} \frac{z^x + a_k^x}{a_k^x - z^x}\,\frac{a_k^x}{\overline{a}_k^x} \prod_{b_k\in D_{\alpha,2R}} \frac{b_k^x - z^x}{z^x + \overline{b}_k^x}\,\frac{\overline{b}_k^x}{b_k^x}, \qquad (2,3)$$

$$g_r(z) = F_r(z)\Phi_{2R}(z,F_r). \qquad (2,4)$$

The function $g_r(z)$, besides satisfying the condition of Lemma 1, satisfies the relation (r fixed)

$$\ln|g_r(r)| = \ln M(r,\infty,f) + \ln|\Phi_{2R}(r,F_r)|. \qquad (2,5)$$

From (2,1), (2,2), (2,3), (2,4), and (2,5) we find, for fixed r,

$$\begin{aligned}\ln^+ M(r,\infty,f) \le (2x)^2 r^{2x} \int_0^R \frac{m(t,f)t^{2x-1}}{(t^{2x}+r^{2x})^2}\,dt \\ + \sum_{|b_k|\le 2R} \ln\left|\frac{r^{2x}+|b_k|^{2x}}{r^{2x}-|b_k|^{2x}}\right| + K_4 x T(2R,f) \\ + K_5 x \cdot \left(\frac{r}{R}\right)^{2x} \{T(4R,f)+T_1(4R,f)\},\end{aligned} \tag{2,6}$$

where $x = \lambda + \varepsilon$, λ is the lower order of $f(z)$ $(\varepsilon > 0)$,

$$T_1(r,f) = \int_0^r \frac{T(s,f)}{s}\,ds$$

and b_k are the poles of $f(z)$.

Estimate (1,2) follows easily from (2,6) (see ([5])). Estimates (1,4), (1,5), and (1,6) are proved in the same way as the corresponding estimates for the quantities of defects of meromorphic functions ([9], [11], [12]).

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Note: Figure translations are in progress. See original paper for figures.