CATEGORIES OF CORRESPONDENCES OVER SEMI-ABELIAN CATEGORIES

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 519.48

MATHEMATICS

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CATEGORIES OF CORRESPONDENCES OVER SEMI-ABELIAN CATEGORIES

(Presented by Academician P. S. Novikov on 22 V 1969)

An axiomatic description is given of the categories of correspondences over semi-abelian categories considered in ([4]). In doing so, first the case of categories without direct sums and products—β-categories—is studied, and then the semi-abelian categories themselves.

For the notions of category theory and the properties of various categories used in this paper without references, see ([1], [4], [5]).

1. We shall call a category $\mathcal{K}$ a **β-category** if it satisfies the following axioms:

SA1(A3). In the category $\mathcal{K}$ there exists a zero object O.

SA2. For any morphism α of $\mathcal{K}$ there exists a cokernel $\operatorname{Coker}\alpha$.

A4.a) For a diagram

$$\xrightarrow{\alpha} \uparrow_{\mu},$$

where μ is a monomorphism, there exists a couniversal square

$$\mu' \uparrow \quad \overset{\alpha}{\underset{\alpha'}{\dashrightarrow}} \quad \uparrow \mu \tag{1}$$

b) moreover, if in the couniversal square (1) α is a normal epimorphism, then α' is also a normal epimorphism.

From axioms SA1 and A4 it follows that:

1) for any morphism α of $\mathcal{K}$ there exists a kernel $\operatorname{Ker}\alpha$;
2) for any two monomorphisms $\mu_i : B_i \to A$, $i = 1, 2$, there exists their intersection $\mu = \mu_1 \cap \mu_2 : B \to A$.

SA3. For any morphism α of $\mathcal{K}$ there exists a canonical decomposition

$$\alpha = (\operatorname{Coker}\operatorname{Ker}\alpha)\,\dot{\alpha}\,(\operatorname{Ker}\operatorname{Coker}\alpha),$$

in which $\dot{\alpha}$ is a bimorphism.

A1. $\mathcal{K}$ is a locally small category on the left, i.e. the subobjects $\bigcup(A)$ of an object $A \in \mathcal{K}$ form a set.

A7(SA4). If in the diagram (2) the rows are exact in the sense of ([4]), then α is an equivalence

$$\begin{array}{ccccccc} 0 & \rightarrow & \cdot & \overset{\mu}{\rightarrow} & \cdot & \overset{\pi}{\rightarrow} & \cdot \rightarrow 0 \\ & & 1\uparrow & & \uparrow\alpha & & \uparrow 1 \\ 0 & \rightarrow & \cdot & \rightarrow & \cdot & & \rightarrow \cdot \rightarrow 0 \\ & & \rho & & \nu & & \end{array} \tag{2}$$

It is easy to see that, for β-categories, properties 1-6 from ([4]) are valid. In addition, in β-categories axiom SA6 always holds, as do such consequences of it as 12-15 (see ([4])), and also the properties:

I. Any pair of morphisms φ_1, φ_2 from $\mathcal{K}$ has a left equalizer.

II. The left equalizer is a normal monomorphism.

For the proof of these properties, axioms A4 and SA3 are used.

Let R be a category with involution (see ([1])), satisfying the axioms:

K1. In R there exists an I-zero object O.

K2. a) If $If \subset Ig$, then $gf^*f \subset g$.

b) If $Bf \supset Bg$, then $gf^*f \supset g$.

K3. For every morphism $u \in R(0, A)$ there exists an injection $m \in R(U, A)$ for which $Bm = u$.

Various properties of such categories were obtained in the papers ([1,3]). In particular, the subcategory $G(R)$ of proper morphisms of the category R satisfies axioms A1, A3, A4, A7 (see ([3])), and for any morphisms $u, v \in R(0, A)$ there exists an intersection $u \cap v \in R(0, A)$.

A morphism $u \in R(0, A)$ will be called **strict** if, for every D-regular morphism $f \in R(A, B)$ and every $v \in R(0, A)$,

$$((u \subset Kf \Rightarrow v \subset Kf) \Rightarrow v \subset u),$$

where $R(A, B)$ is the set of morphisms from an object $A \in R$ to an object $B \in R$. We note that a morphism equal to Kf for some D-regular morphism f is always strict.

K7^0. For every strict morphism $u \in R(0, A)$ there exists a projection $p \in R(A, U)$ for which $Kp = u$.

K8. For every morphism $v \in R(0, A)$ there exists a minimal strict morphism $u \in R(0, A)$ for which $u \supset v$.

K9. If an injection m is a left equalizer in the category $G(R)$, then Bm is a strict morphism.

In a β-category $\mathcal{K}$ one can define a correspondence between objects $A, B \in \mathcal{K}$ by means of a construction analogous to the construction of correspondences in quasi-exact and γ-categories (see ([2,3])), and show, as in the paper ([3]), that the following is true.

Theorem 1. For an arbitrary β-category $\mathcal{K}$ there exists an extension to a category with involution $R(\mathcal{K})$, satisfying axioms K1–K3, K7^0–K9, and the category $\mathcal{K}$ is isomorphic to the subcategory $G(R(\mathcal{K}))$ of proper morphisms of the category $R(\mathcal{K})$.

Let R be a category with involution satisfying axioms K1, K2.

Lemma. If m, n are injections, and Bm, Bn are strict morphisms, then Bmn is a strict morphism $(m \in R(A,B),\ n \in R(B,C))$.

With the help of this lemma and axioms K7^0–K9 it is verified that the category $G(R)$ satisfies axioms SA2 and SA3, i.e.

Theorem 2. If a category with involution R satisfies axioms K1–K3, K7^0–K9, then its subcategory of proper morphisms $G(R)$ is a β-category.

Theorem 3. Between β-categories and categories with involution satisfying axioms K1–K3, K7^0–K9, there exists a one-to-one correspondence up to equivalence of categories.

The proof is carried out in the same way as for γ-categories (see ([3])), relying on Theorems 1, 2 and the following

Proposition. If categories with involution R and R' satisfy axioms K1–K3, K7^0–K9, then:

1) an I-functor (see ([1])) $Q : R \to R'$ induces an exact functor

$$BQ : G(R) \to G(R'),$$

preserving intersections of subobjects;

2) every exact functor $F : G(R) \to G(R')$, preserving intersections of subobjects, can be uniquely extended to an I-functor

$$EF : R \to R'.$$

2. Let the β-category $\mathcal{K}$ satisfy axiom A8. For any objects $A, B \in \mathcal{K}$ there exists a direct product

$$A \times B \in \mathcal{K}.$$

Then, just as in the case of abelian categories (see ([1])), one can show that the β-category $\mathcal{K}$ will be additive (in the sense of Grothendieck ([6])), i.e. it satisfies axiom SA5 and is semiabelian (see ([4])). In this case, instead of axiom A8 one may require the existence in the category $\mathcal{K}$ of a couniversal square for an arbitrary diagram

or the fulfillment of the axiom dual to A8:

A8′. For any objects $A, B \in \mathcal{K}$ there exists a direct sum (free product)

$$A \oplus B \in \mathcal{K}.$$

Let a category with involution R satisfy axiom K1. We introduce some additional conditions (see (1)):

K4. In the category R, for any pair of morphisms $f, g \in R(A, B)$, $A, B \in R$, there exist:

a) their intersection $f \cap g \in R(A, B)$;

b) their union $f \cup g \in R(A, B)$.

Remark. The intersection of strict morphisms (see § 1) is a strict morphism.

K5. Let $f \in R(A, B)$, $g_1, g_2 \in R(B, C)$. Then:

a) from $If \subset Kg_1$ it follows that

$$f(g_1 \cap g_2) \supset fg_1 \cap fg_2;$$

b) from $Bf \supset Dg_1$ it follows that

$$f(g_1 \cup g_2) \subset fg_1 \cup fg_2.$$

We note that the relations

$$f(g_1 \cap g_2) \subset fg_1 \cap fg_2$$

and

$$f(g_1 \cup g_2) \subset fg_1 \cup fg_2$$

follow from the monotonicity of multiplication and from the properties of intersection and union of morphisms.

K6. For any objects $A_1, A_2 \in R$ there exist:

a) an object $P \in R$ and proper morphisms $g_i : P \to A_i$, $i = 1, 2$, for which

$$g_1^* g_2 = \Omega_{A_1 A_2};$$

b) an object $S \in R$ and proper morphisms $h_i : A_i \to S$, $i = 1, 2$, for which

$$h_1 h_2^* = \omega_{A_1 A_2}.$$

Theorem 4. 1) In a category with involution R satisfying axioms K1–K3, $K7^0$–K9, the conditions K4a)–K6a) and K4b)–K6b) are equivalent. 2) If a category with involution R satisfies axioms K1–K9, then its subcategory $G(R)$ is semi-abelian.

The proof is carried out analogously to that of D. Puppe ([1]) for the abelian case, relying here on the results of the preceding paragraph.

In a semi-abelian category $\mathcal{K}$ one can, in the usual way, introduce correspondences as subobjects of direct products and consider the category of correspondences $R(\mathcal{K})$ (see ([4])).

Theorem 5. 1) The category of correspondences $R(\mathcal{K})$ over a semi-abelian category $\mathcal{K}$ satisfies axioms K1–K9. 2) Between semi-abelian categories and categories with involution satisfying axioms K1–K9 there is a one-to-one correspondence up to equivalence of categories.

In ([4]) it is shown that the category $R(\mathcal{K})$ satisfies axioms K1–K6 and that the category $\mathcal{K}$ is isomorphic to the subcategory $G(R(\mathcal{K}))$ of proper morphisms of the category $R(\mathcal{K})$. Verifying axioms $K7^0$–K9, we obtain the first assertion of the theorem, and, using the assertion from § 1, also the second.

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Received
7 V 1969

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Note: Figure translations are in progress. See original paper for figures.