EQUATIONS WITH MONOTONE AND POTENTIAL OPERATORS IN BANACH SPACES

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EQUATIONS WITH MONOTONE AND POTENTIAL OPERATORS IN BANACH SPACES

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1. Nonlinear equations with monotone and potential operators were studied by the authors in a number of papers ([1–8]). Papers ([1–3]) were the first in which nonlinear equations with monotone potential operators were studied, while in papers ([4–5]) nonlinear equations with monotone operators whose potentiality was not assumed were first studied. Later the theory of such equations was developed in the works of F. Browder, G. Minty, and other authors (see, for example, ([9]), where a bibliography is given).

In the present paper new theorems on the existence and uniqueness of a solution of the equation

$$x = SF(x), \tag{1}$$

are established, where $F(x)$ is a nonlinear operator from the Banach space $\mathcal{X}$ into the Banach space $\mathcal{Y}$, and S is a linear operator from $\mathcal{Y}$ into $\mathcal{X}$. Equations of the form (1) have been studied by many authors. However, in all works prior to ([6]) the operator S was assumed bounded. Only in papers ([6,7]) were assertions on solutions of equation (1) first established without assuming boundedness of the operator S.

In the first part of the present paper the operator $F(x)$ is assumed to be potential and S bounded or unbounded, while in the second part the operator $F(x)$ is assumed monotone.

2. Let the Banach space E satisfy condition (α_1): there exists a Hilbert space H such that the linear set $H_0 \subset H \cap E^*$ is dense both in H and in E^*, and the value of the linear functional $y \in E^*$ on a vector $x \in E$, i.e. $\langle y, x\rangle$, coincides with the scalar product (y, x) in H for $x, y \in H$.

Examples of such spaces are the spaces L^p, L^2, L^q $(p^{-1} + q^{-1} = 1,\ p \geq 2)$ on a set of finite or infinite measure.

We shall say that a linear operator B has property (β_1) if the following conditions are fulfilled: B is a bounded operator from E^* into E^{**} such that its restriction

B_0 to H_0 is a symmetric operator admitting a self-adjoint positive extension B_H with domain of definition $D(B_H) \supset H_0$. Note that boundedness of B_H does not follow from property (β_1). Let us also note that if E is reflexive, then an operator B with property (β_1) acts from E^* into E.

Theorem 1. *Suppose condition (α_1) is fulfilled, the operator B has property (β_1), and $S = B^*$. Suppose further that $F(x)$ is a potential operator defined on E^{**}, whose potential $f(x)$ is weakly lower semicontinuous from above and satisfies the inequality*

$$f(x) \le -\omega(\|x\|), \qquad x \in E^{**}, \tag{2}$$

where $\omega(t) < 0$ for $t \ge 0$, and for some positive R and $\delta = \delta(R)$ the relation

$$\inf_{[0,a]} R^{-2}\omega(t) \ge -\frac{1}{2} + \delta, \qquad R^2\delta > -f(0), \qquad a = R\|B\|^{1/2}.$$

Then equation (1) has a solution in E^{**}.

Remark 1. If in the hypotheses of Theorem 1 the space E is reflexive, then the operator $S = B$.

Let the Banach space E satisfy **condition** (α_2): there exists a Hilbert space H such that the linear set $H_0 \subset H \cap E$ is dense both in H and in E, and the value of the linear functional $y \in E^*$ on the vector $x \in E$, i.e. $\langle y, x\rangle$, coincides with the scalar product (y, x) in H for $x, y \in H$. Further, we shall say that the linear operator B has **property** (β_2) if B is a bounded operator from E into E^* such that its restriction to H_0 is a symmetric operator admitting a self-adjoint positive extension.

Theorem 2. *Let condition (α_2) be fulfilled, let the operator B have property (β_2), let $S = B^*$, and let $F(x)$ be a potential operator defined on E^*, whose potential $f(x)$ satisfies the hypotheses of Theorem 1 for $x \in E^*$. Then equation (1) has a solution in E^*.*

Theorem 3. *Let the reflexive Banach space E satisfy condition (α_1), let the operator B have property (β_1), and let $F(x)$ be a demicontinuous potential operator whose potential is weakly upper semicontinuous, and moreover*

$$(F(x), x) \le a(x, x) + b(x, x)^\gamma + c \quad (0 < \gamma < 1,\ a > 0,\ b > 0,\ c > 0,\ x \in D),$$

where $D = \{x \in H_0 : B_H^{1/2} x \in H\}$ is dense in H and $a\|B\| < 1$.

Then equation (1), where $S = B$, is solvable in E.

3. We shall say that a linear operator B from E^* into E has **property** (β_3) if B is a closed operator with dense domain of definition $D(B)$, whose restriction B_0 to a set $\widetilde{H}_0 \subset D(B) \cap H_0$ dense in H (see condition (α_1)) is

a symmetric positive operator admitting a self-adjoint positive extension B_H.

For what follows we introduce the notion of a generalized solution of equation (1). In doing so we shall assume that the set $\widetilde{H}_0$ is maximal. Let E_0 be the closure of $\widetilde{H}_0$ in the metric

$$|x| = \|x\|_{E^*} + \|B^* x\|_E \quad (x \in \widetilde{H}_0).$$

Definition 1. Suppose that for some vector $x_0 \in E$, where E is a reflexive Banach space, the equality

$$\langle h, x_0 \rangle = \langle B_* h, F(x_0) \rangle$$

holds, where h is an arbitrary vector from the set E_0, and moreover $B^*[E_0]$ is dense in E. Then we shall say that x_0 is a **generalized solution** of the equation

$$x = BF(x). \tag{3}$$

Denote by U the closure of $D(B)$ in the metric

$$|x| = \|x\|_{E^*} + \|Bx\|_E \quad (x \in D(B)).$$

Definition 2. If there exists a linear set G, dense both in H and in U, then the operator B will be called G**-perfect**.

Lemma 1. *If B^* is a $\widetilde{H}_0$-perfect operator, then every generalized solution of equation (3) will be an exact solution of this equation.*

Let

$$D_0 = \{h \in D(B_H^{1/2}) : B_H^{1/2} h \in E\}.$$

Theorem 4. *Let the reflexive space E satisfy condition (α_1), and let the operator B have property (β_3). Suppose further that the sets D_0 and $B[\widetilde{H}_0]$ are dense respectively in H and E.*

Then, if the potential $f(x)$ of the operator $F(x)$ is weakly upper semicontinuous and satisfies the inequality

$$f(x) \leq -\omega(\|x\|_E), \qquad \lim_{t \to +\infty} \omega(t) = +\infty,$$

where $\omega(t)$ is bounded below for $t \geq 0$, then equation (3) has a generalized solution in E.

We shall say that a linear operator B from E^* into E has **property** (β_4) if B is a closed operator with dense domain of definition, whose restriction B_0 to a set $\widetilde{H}_0 \subset D(B) \cap H_0$ dense in H (see (α_1)) is a symmetric operator admitting a self-adjoint extension—

B_H. Let

$$B_H^+ = \frac{1}{2}(|B_H| + B_H), \qquad B_H^- = \frac{1}{2}(|B_H| - B_H), \qquad A = (B_H^+)^{1/2} + (B_H^-)^{1/2},$$

$$C = (B_H^+)^{1/2} - (B_H^-)^{1/2}, \qquad D_0(A) = \{h \in D(A) : Ah \in E\},$$

$$D_0(C) = \{h \in D(C) : Ch \in E\}.$$

Definition 3. A nonlinear operator $F(x)$ from E into E^* will be called **hemicontinuously integrable** if the function $(F(tx), x)$ is summable on $(0, 1)$ for every $x \in E$.

Theorem 5. *Suppose the following conditions are satisfied:*

1) *The reflexive Banach space E satisfies condition (α_1), and the operator B has property (β_4).*
2) *The sets $D_0(A)$, $D_0(C)$, and $B[\widetilde{H}_0]$ are dense respectively in H and E.*
3) *The spectrum of the operator B_H lies outside $(0, m)$, where m is some positive number.*
4) *The hemicontinuously integrable potential operator $F(x)$, defined in E, satisfies the inequality*

$$(F(x) - F(y),\, x - y) \geq \frac{1+\alpha}{m}\|x - y\|_H^2 + \beta\|x - y\|_E^2$$

$$(x, y \in D), \qquad D = \{x \in \widetilde{H}_0 : Ax \in H\} \text{ is dense in } H, \quad \alpha > 0, \quad \beta > 0.$$

Then equation (3) has a generalized solution in E.

Remark 2. If B^* is an $\widetilde{H}_0$-perfect operator, then the generalized solution is unique and exact.

4. Here and in item 5 we shall consider equations (1) and (3) under the assumption that $F(x)$ is a monotone operator (without the assumption of its potentiality) and B is a linear operator.

Theorem 6. *Let the reflexive Banach space E satisfy condition (α_1), let the operator B have property (β_1), and let $F(x)$ be a hemicontinuous operator from E into E^*, satisfying the inequality*

$$(F(x) - F(y),\, x - y) \leq 0 \qquad (x, y \in E). \tag{4}$$

Then equation (3) has a unique solution belonging to the space E.

The assertion of Theorem 6 remains valid if inequality (4) is replaced by the inequality

$$(F(x) - F(y),\, x - y) \leq \gamma(\|x-y\|_E)\|x-y\|_E,$$

where

$$\sup_{t\in[0,Mr]} \frac{\gamma(t)}{r} \leq \frac{1}{M} - \alpha(r), \qquad \gamma(t) > 0, \quad M^2 = \|B\|, \quad \alpha(r) > 0, \quad r\alpha(r) \to +\infty$$

as $r \to +\infty$.

Remark 3. If, in the hypotheses of Theorem 6, inequality (4) is replaced by the inequalities

$$(F(x) - F(y),\, x - y) \leq \gamma_1(\|x-y\|_E)\|x-y\|_E,$$

$$(F(x), x) \leq \gamma_2(\|x\|_E)\|x\|_E,$$

where $\gamma_i(t) \geq 0$ $(i = 1, 2)$ for $t \geq 0$ and

$$\sup_{t\in\sigma} \frac{\gamma_1(t)}{r} \leq \frac{1}{M}, \qquad \sup_{t\in\sigma} \frac{\gamma_2(t)}{r} \leq \frac{1}{M} - \alpha(r), \qquad \sigma = [0, Mr],$$

$$M^2 = \|B\|, \qquad r\alpha(r) \to +\infty \quad \text{as } r \to +\infty,$$

then equation (3) has a solution belonging to E.

Remark 4. If, in the hypotheses of Remark 3,

$$\sup_{t\in\sigma} \frac{\gamma_1(t)}{r} < \frac{1}{M}, \qquad \sigma = [0, Mr],$$

then equation (3) has a unique solution in E.

Theorem 7. *Let the space E satisfy condition (α_2), let the operator B have property (β_2), and let $F(x)$ be a hemicontinuous operator from E^* into E^{**} such that*

$$(F(x) - F(y), x - y) \leq 0 \qquad (x, y \in E^*).$$

Then equation (1) for $S = B^*$ has a unique solution in E^*.

We note that, with respect to this theorem, remarks analogous to Remarks 3 and 4 are valid.

5. Let B be a linear operator having property (β_5): B is a bounded operator from E^* into E such that its restriction B_0 to H_0 (see (α_1)) is a symmetric operator admitting a self-adjoint extension B_H.

We note that from property (β_5) boundedness of the operator B_H does not follow. Let

$$B_H^+ = \frac{1}{2}\left(|B_H| + B_H\right), \quad B_H^- = \frac{1}{2}\left(|B_H| - B_H\right), \quad A = (B_H^+)^{1/2} + (B_H^-)^{1/2}.$$

Theorem 8. Let the following conditions be fulfilled:

1) The reflexive space E satisfies condition (α_1), and the operator B has property (β_5).
2) The spectrum of the operator B_H lies outside the interval $(0, m)$, where m is a certain positive number.
3) The demicontinuous operator $F(x)$, acting from E into E^*, satisfies the inequality

$$(F(x) - F(y),\, x - y) \geq \frac{1+\alpha}{m}\|x - y\|_H^2 \quad (x, y \in D), \tag{5}$$

where $D = \{x \in H_0 : Ax \in H\}$ is dense in H and $\alpha > 0$.

Then equation (3) has a unique solution in E.

Remark 5. We note that under the conditions of Theorem 8 it is sufficient to require of the operator $F(x)$ that it be hemicontinuous and locally bounded, since, when inequality (5) is fulfilled, an analogue of Kato' s theorem ([10]) holds.

Remark 6. If $E \subset H \subset E^*$, then in inequality (5) one may put $D = E$. In this case the requirement of demicontinuity of the operator $F(x)$ can be replaced by the requirement of its hemicontinuity.

6. The proof of the formulated theorems uses new propositions on the square root of linear operators, of which we present the following.

Theorem 9. Let the space E satisfy condition (α_1), let the operator B have property (β_3), and let D_0 be dense in H. Then the positive square root $B_H^{1/2}$ can be extended from $\widehat{H}_0$ to a closed operator T acting from E^* into H, and moreover $T^*Tx = Bx$ $(x \in \widehat{H}_0)$.

Theorem 10. Let the conditions of Theorem 9 be fulfilled and let the operator B be $\widehat{H}_0$-perfect. Then $T^*Tx = Bx$ $(x \in D(B))$.

Theorem 11. Let the space E satisfy condition (α_1), let the operator B have property (β_4), and let the sets $D_0(A)$, $D_0(C)$ be dense in H. Then the operators

A and C can be extended from $\widehat{H}_0$ to, respectively, closed operators V and W from E^* into H, and moreover $V^*Wx = W^*Vx = Bx$ $(x \in \widehat{H}_0)$.

Theorem 12. Let the conditions of Theorem 11 be fulfilled and let the operator B be $\widehat{H}_0$-perfect. Then for any vector $x \in D(B)$ the relations $V^*Wx = W^*Vx = Bx$ are valid.

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Note: Figure translations are in progress. See original paper for figures.