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MATHEMATICS

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ON LARGE INDUCTIVE DIMENSION

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§ 1. In paper (1), for a topological space* X, the notion of cofinal approximative dimension ΔX is introduced, and it is proved that the space X has cofinal approximative dimension $\Delta X \leq n$ if and only if it is a closed, at most $(n+1)$-fold image of a completely zero-dimensional** space Y. In this case the space X is paracompact. In the same paper (1) it is shown that always $\operatorname{Ind} X \leq \Delta X$.

Thus, the cofinal approximative dimension of a space allows one to estimate from above the large inductive dimension of this space. In this note, in particular, we shall be interested in the following general question:

Let X and Y be two bicompacta. The question is whether one may assert that

$$\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y. \tag{1}$$

This question, in its full generality, has not yet been resolved.*** However, the following basic result holds.

Theorem 1. *Let $X_1, X_2, \dots, X_k$ be arbitrary bicompacta. Then*

$$\operatorname{Ind}\left(\prod_{i=1}^{k} X_i\right) \leq \sum_{i=1}^{k} \Delta X_i.$$

We supplement this theorem with the following proposition.

Theorem 2. *Let $X_1, \dots, X_k$ all be bicompacta, with the possible exception of one which is paracompact. Then also*

$$\operatorname{Ind}\left(\prod_{i=1}^{k} X_i\right) \leq \sum_{i=1}^{k} \Delta X_i.$$

* In this note only normal spaces are considered.
** A T_1-space Y is called completely zero-dimensional if it is regular and into each of its open covers one can inscribe a cover consisting of pairwise disjoint open sets.
*** In this direction there are some results in papers (2, 3), where inequality (1) is proved in the case when the product $X \times Y$ is totally normal. The requirement of total normality is rather restrictive, since simple examples are known of completely normal bicompacta whose square is not hereditarily normal. If one considers inequality (1) in the case when the number of factors is greater than two, then recall M. Katětov' s result that the cube $X \times X \times X$ of a bicompactum X is hereditarily normal if and only if X is metrizable. In my paper (5) a class of bicompacta is determined, denoted by the letter B, and inequality (1) is proved for an arbitrary finite number of factors in the case when the factors belong to this class of bicompacta B. From my theorem, in particular, it follows that the inequality $\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y$ holds in the case when X and Y are totally normal bicompacta. I note that in my theorem no conditions are imposed on the space $X \times Y$, unlike in the results mentioned above.

From Theorem 1 it follows

Corollary 1. Let X and Y be bicompacts and $\dim X = \Delta X = n$, $\dim Y = \Delta Y = m$. If one of the bicompacts X or Y is dimensionally full-valued (in the sense of the dimension dim), then

$$\dim(X \times Y) = \operatorname{ind}(X \times Y) = \operatorname{Ind}(X \times Y) = n + m.$$

Remark 1. Since for the ordered continuum X, $\dim X = \Delta X = 1$, it follows from Corollary 1 that the theorem obtained by me in ([7]) follows: for the product of a finite number of ordered continua, the dimensions dim, ind, and Ind coincide.

§ 2. Let P be some topological condition that can be imposed on open covers of a topological space X. For example: P_1 –point-binaryness*, P_2 –binaryness**, P_3 –finiteness, P_4 –local finiteness, P_5 –the property of being an open cover of the topological space X, etc. We introduce the following notion $\operatorname{Ind}_P X$ of the dimension of a topological space X.

Definition 1. For the empty set $\emptyset$, and only for it, we set $\operatorname{Ind}_P \emptyset = -1$.

Suppose the spaces X having dimension $\operatorname{Ind}_P \leq n-1$ have already been defined. Then the space X has $\operatorname{Ind}_P X \leq n$ if in every open cover ω satisfying the condition P one can inscribe such a disjoint system $\eta = \{V_\alpha\}$ of open sets that

$$\operatorname{Ind}_P\Big(X \setminus \bigcup_\alpha V_\alpha\Big) \leq n - 1.$$

Taking different conditions P, we shall obtain different inductive invariants of the topological space X, some of which were known earlier. For example:

$\operatorname{Ind}_{P_1} X = \operatorname{ind} X$, $\operatorname{Ind}_{P_2} X = \operatorname{Ind} X$, $\operatorname{Ind}_{P_3} X = \operatorname{Dind} X$ (this dimension was previously proposed by A. V. Arhangel' skii (see ([8]))).

The dimension defined by means of locally finite covers will be denoted by $\operatorname{Ind}_T X$ and called the total dimension of the space X.

In ([8]) it was proved that for an arbitrary normal space X the inequality $\operatorname{Dind} X \geq \operatorname{Ind} X$ holds.

Lemma 1. Let the space X be paracompact. Then

$$\operatorname{Ind}_T X \geq \operatorname{Ind} X.$$

Remark 2. Total dimension is monotone with respect to closed subsets in a paracompact space; in a non-paracompact space this is apparently not so.

Remark 3. For every bicompact X the equality

$$\operatorname{Dind} X = \operatorname{Ind}_T X$$

holds.

Lemma 2. Suppose that in the space X the sum theorem holds for large inductive dimension. Then

$$\operatorname{Dind} X = \operatorname{Ind} X.$$

Corollary 2. If the space X is normal and $\operatorname{Ind} X \leq 1$, then $\operatorname{Dind} X = \operatorname{Ind} X$.

Definition 2. We shall say that a hereditarily normal space X belongs to the class Π of spaces if, for the space X, the large inductive dimension is monotone with respect to open subsets.

* A cover is called point-binary if it consists of a neighborhood of some point and its complement.

** A cover is called binary if it consists of a neighborhood of some closed set and its complement.

In paper ([9]) it is proved that in such spaces the sum theorem holds for closed subsets. Therefore the following is also true.

Corollary 3. Let the hereditarily normal space X belong to the class Π. Then

$$\operatorname{Dind} X = \operatorname{Ind} X.$$

The proof of the following Lemma 3 is contained in the proof of Lemma 5 of paper ([10]).

Lemma 3. Let the hereditarily normal space X, belonging to the class Π, be paracompact. Then

$$\operatorname{Ind}_T X = \operatorname{Dind} X = \operatorname{Ind} X.$$

Theorem 3. Let the space X have combinatorial approximation dimension not exceeding n. Then

$$\operatorname{Ind}_T X \leq n.$$

Theorem 4. Let $X_1, X_2, \dots, X_k$ be bicompacts. Then

$$\operatorname{Dind}\left(\prod_{i=1}^{k} X_i\right) \leq \sum_{i=1}^{k} \Delta X_i.$$

Theorem 5. Let $X_1, \dots, X_k$ all, with the possible exception of one, be bicompacts that are paracompact. Then

$$\operatorname{Ind}_T\left(\prod_{i=1}^{k} X_i\right) \leq \sum_{i=1}^{k} \Delta X_i.$$

Theorem 6. Let $X_1, \dots, X_k$ be bicompacts of class (in particular, one-dimensional, perfectly normal, or dyadic; see ([4])). Then

$$\operatorname{Dind}\left(\prod_{i=1}^{k} X_i\right) \leq \sum_{i=1}^{k} \operatorname{Ind} X_i = \sum_{i=1}^{k} \operatorname{Dind} X_i.$$

Corollary 4. Let $X_1, \dots, X_k$ be one-dimensional bicompacts. Then

$$\operatorname{Dind}\left(\prod_{i=1}^{k} X_i\right) = \dim\left(\prod_{i=1}^{k} X_i\right) = \operatorname{ind}\left(\prod_{i=1}^{k} X_i\right) = \operatorname{Ind}\left(\prod_{i=1}^{k} X_i\right) = k.$$

In order to formulate the next theorem of this section, let us recall some known definitions:

a) ([6]). A mapping $f : X \to Y$ will be called a $(0, \Omega)$-mapping, where $\Omega = \{\omega\}$ is a system of open covers of the space X, if for every cover $\omega \in \Omega$ and every point $y \in Y$ there exists a neighborhood Oy such that the set $f^{-1}(Oy)$ decomposes into a disjoint system of open subsets of X inscribed in the cover ω.

b) ([10]). A mapping $f : X \to Y$ is called (Ω, Θ)-uniformly zero-dimensional, where $\Omega = \{\omega\}$ is some system of covers of the space X, and $\Theta = \{\theta\}$ is some system of covers of the space Y, if for every cover $\omega \in \Omega$ there exists a cover $\theta \in \Theta$ such that the inverse image of each element of θ decomposes into a disjoint system of open subsets of X inscribed in the cover ω.

Theorem 7. Let P_1 and P_2 be topological conditions on open covers of spaces. Let the mapping $f : X \to Y$ of the space X onto the space Y be (Ω, Θ)-uniformly zero-dimensional, where $\Omega = \{\omega\}$ is the system of all open covers of the space X satisfying condi-

condition P_1, and $\Theta = \{\theta\}$ is the system of all open covers of the space Y satisfying condition P_2. Then

$$\operatorname{Ind}_{P_1} X \leq \operatorname{Ind}_{P_2} Y.$$

Remark 4. A $(0, \Omega)$-mapping of the space X onto the space Y is (Ω, Θ)-uniformly zero-dimensional, where Θ is the system of all open covers of the space Y.

Corollary 5. Let a normal space X be $(0, \Omega)$-mapped onto a paracompact Y with $\operatorname{Ind} Y \leq 1$, where $\Omega = \{\omega\}$ is the system of all finite (locally finite) open covers of the space X. Then

$$\operatorname{Ind} X \leq \operatorname{Dind} X \leq 1 \qquad (\operatorname{Ind}_T X \leq 1).$$

Corollary 6. If a mapping f of a normal space X onto a paracompact hereditarily normal space Y, belonging to class Π, is a $(0, \Omega)$-mapping with respect to the system of all finite covers of the space X. Then

$$\operatorname{Ind} X \leq \operatorname{Dind} X \leq \operatorname{Ind} Y = \operatorname{Ind}_T Y.$$

Corollary 6 is a reformulation, under more general assumptions, of Theorem 4 from [(10)] and Theorem 14 from [(6)].

Theorem 8. Let $X_1, \dots, X_k$ be bicompacts admitting zero-dimensional mappings onto bicompacts from class B (in particular, one-dimensional, perfectly normal, and Dowker spaces). Then

$$\operatorname{Ind}\left(\prod_{i=1}^{k} X_i\right) \leq \sum_{i=1}^{k} \operatorname{Ind} X_i.$$

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Note: Figure translations are in progress. See original paper for figures.