# ON THE EXISTENCE OF INVARIANT TORI IN A NEIGHBORHOOD OF AN EQUILIBRIUM STATE OF A SYSTEM OF DIFFERENTIAL EQUATIONS

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**Abstract**

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# ON THE EXISTENCE OF INVARIANT TORI IN A NEIGHBORHOOD OF AN EQUILIBRIUM STATE OF A SYSTEM OF DIFFERENTIAL EQUATIONS

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In the paper (1) a system of second-order differential equations was considered,

$$\ddot{x}_k + \lambda_k^2 x_k = f(x_1, \dot{x}_1^2, \dots, x_n, \dot{x}_n^2) \quad (k = 1, \dots, n), \tag{1}$$

where $f_k$ are series in powers of $x_j, \dot{x}_j^2$, containing in their expansions no terms of degree lower than the second. It was shown that if

$$k_1\lambda_1 + \dots + k_n\lambda_n \neq 0$$

for any integers $k_1, \dots, k_n$ not all equal to zero simultaneously, then in any neighborhood of the origin of coordinates of the phase space the system (1) has invariant $n$-dimensional torus-like surfaces. To prove the existence of these surfaces, the process of constructing successive approximations proposed by A. N. Kolmogorov (Newton' s method (2,3)) was used. The possibility of carrying out this process for system (1) was based on the fact that at each step the coefficients of the expansion of the right-hand sides of the resulting systems turned out to be purely imaginary.

In fact, in order to carry out the indicated constructions it is sufficient that only some of these coefficients turn out to be purely imaginary. In the present work we consider arbitrary systems of differential equations and apply Newton' s method. It is assumed that at each step the corresponding coefficients turn out to be purely imaginary (condition A below). Then, using the arguments and estimates of (1), one can prove the convergence of the successive approximations and obtain an analogous result (Theorem 1 below) for arbitrary systems as well. In contrast to (1), we consider the case in which the right-hand sides of the system depend periodically on time, which leads to certain special features in the construction of the successive approximations. Below the precise assumptions and results are formulated, and a description is given of an arbitrary step in Newton' s method.

1°. Consider a system of differential equations

$$\dot{x}_k = X_k(x_1, y_1, \dots, x_n, y_n, t),$$

$$\dot{y}_k = Y_k(x_1, y_1, \dots, x_n, y_n, t) \quad (k = 1, \dots, n), \tag{2}$$

where $X_k, Y_k$ are series in powers of $x_j, y_j$ $(j = 1, \dots, n)$ with real continuous coefficients periodic in $t$, converging in some neighborhood of the origin of coordinates and vanishing for $x_j = y_j = 0$ $(j = 1, \dots, n)$. Everywhere below the period is assumed equal to $2\pi$. Let the characteristic exponents of the first-approximation system be purely imaginary, equal to $\pm\lambda_k i$, and satisfy the condition

$$k_1\lambda_1 + \dots + k_n\lambda_n \neq p$$

for any integers $k_j, p$ not all equal to zero simultaneously.

Under the assumptions made, for any natural number $N$ the system (2), by means of the transformation

$$x_k = U_k(u_j, v_j, t), \qquad y_k = V_k(u_j, v_j, t) \quad (k, j = 1, \dots, n),$$

where $U_k, V_k$ are polynomials in $u_j, v_j$ of degree not exceeding $2N$ with periodic coefficients, can be reduced to the form

$$\begin{aligned} \dot{u}_k &= i\lambda_k u_k + u_k G_k(u_{jv}j) + F_k(u_j, v_j, t), \\ \dot{v}_k &= -i\lambda_k v_k + v_k \overline{G}_k(u_{jv}j) + \overline{F}_k(u_j, v_j, t) \qquad (k, j = 1, \dots, n), \end{aligned} \tag{3}$$

where $G_k$ and $\overline{G}_k$ are polynomials of degree not exceeding $N-1$ in the products $u_{jv}j$, with constant coefficients, while $F_k$ and $\overline{F}_k$ are series in powers of $u_j, v_j$ with periodic coefficients whose expansions contain no terms of dimension lower than $2N+1$ [4]. Moreover, if $x_k, y_k$ are real, then $u_k$ and $v_k$ are complex conjugates: $u_k = \overline{v}_k$ $(k = 1, \dots, n)$. In system (3), and below, by $\overline{W}(u_j, v_j, t)$ we mean the function complex conjugate to $W(u_j, v_j, t)$ in the domain of reality of $x_k, y_k$. We assume that the coefficients of the polynomials $G_k$ are purely imaginary, i.e.

$$G_k(u_{jv}j) = -\overline{G}_k(u_{jv}j) \equiv iH_k(u_{jv}j) \qquad (k, j = 1, \dots, n),$$

where $H_k$ is a polynomial with real coefficients, and that if $J$ is the Jacobian of the functions $H_k(z_1, \dots, z_n)$ at $z_1 = \dots = z_n = 0$, then

$$J \neq 0. \tag{4}$$

Let $\varepsilon > 0$ be sufficiently small. We shall consider system (3) in the domain $|u_j| < \varepsilon,\ |v_j| < \varepsilon$. Choose real constants $\mu_1, \dots, \mu_n$ so that the inequality

$$\left|\sum_{j=1}^{n} k_j \mu_j - p\right| > \varepsilon^3 \left(\sum_{j=1}^{n} |k_j|\right)^{-n-2} \tag{5}$$

holds for all integers $k_j, p$ not simultaneously equal to zero. Condition (5) is satisfied by most points of the $\varepsilon^2$-neighborhood of the point $(\lambda_1, \dots, \lambda_n)$ (see [3], where the case $p = 0$ is considered).

We shall transform system (3) by means of Newton' s method. We shall regard system (3) as the first approximation. Introduce the sequence $N_s$ of natural numbers defined by the equalities $N_1 = N,\ N_{s+1} = 2N_s$.

Suppose that, as the result of the first $s-1$ steps, we have obtained the system

$$\begin{aligned} \dot{u}_{ks} &= i\mu_k u_{ks} + iu_{ks}[(\lambda_k - \mu_k) + H_{ks}(z_{js})] + \Phi_{ks}(u_{js}, v_{js}, t), \\ \dot{v}_{ks} &= -i\mu_k v_{ks} - iv_{ks}[(\lambda_k - \mu_k) + H_{ks}(z_{js})] + \overline{\Phi}_{ks}(u_{js}, v_{js}, t) \qquad (k, j = 1, \dots, n), \end{aligned} \tag{6}$$

where

$$H_{ks}(z_{js}) = \sum_{\sigma=1}^{N_s-1} a_{ks}^{(\sigma_1, \dots, \sigma_n)} z_{1s}^{\sigma_1} \cdots z_{ns}^{\sigma_n},$$

$$\Phi_{ks}(u_{js}, v_{js}, t) = \sum_{m=2N_s+1}^{\infty} M_{ks}^{(k_1, l_1, \dots, k_n, l_n)}(t) u_{1s}^{k_1} \cdots v_{ns}^{l_n},$$

$$z_{js} = u_{js} v_{js}; \qquad \sigma = \sigma_1 + \dots + \sigma_n, \quad \sigma_j \geq 0; \qquad m = k_1 + l_1 + \dots + k_n + l_n,$$

$$k_j \geq 0, \quad l_j \geq 0.$$

Here $a_{ks}^{(\sigma_j)}$ and $M_{ks}^{(k_j, l_j)}$ are functions of $z_{js}$, analytic in a domain of the form

$$r_{js}^2 - \delta_s < |z_{js}| < r_{js}^2 + \delta_s \qquad (j = 1, \dots, n)$$

with sufficiently small positive $\delta_s$. Moreover, the coefficients of the expansions of the functions $a_{ks}^{(\sigma_j)}$ are real and constant, while the coefficients of the expansions of the functions $M_{ks}^{(k_j,l_j)}$ are periodic in $t$. Obviously, for $s=1$ these assumptions are satisfied. We make one more assumption:

condition A. The mean values of the coefficients of the functions $M_{ks}^{(k_j,l_j)}$, for which $k_j = l_j + \delta_{kj}$ for $2N_s+1 \le m \le 4N_s$ ($k,j=1,\dots,n$; $\delta_{kj}$ is the Kronecker symbol), are purely imaginary numbers.

The $s$-th step consists in reducing system (6), by means of the transformation

$$\begin{aligned} u_{k,s+1} &= u_{ks} + \varphi_{ks}(u_{js}, v_{js}, t), \\ v_{k,s+1} &= v_{ks} + \overline{\varphi}_{ks}(u_{js}, v_{js}, t) \qquad (k,j=1,\dots,n), \end{aligned}$$

where

$$\varphi_{ks}(u_{js}, v_{js}, t) = \sum_{m=2N_s+1}^{4N_s} L_{ks}^{(k_1,\dots,l_n)}(t) u_{1s}^{k_1} \cdots v_{ns}^{l_n};$$

$L_{ks}^{(k_j,l_j)}$ are functions of $z_{js}$, analytic in the domain

$$|(\lambda_k + \mu_k) + H_{ks}(z_{1s}, \dots, z_{ns})| < \Delta_s \qquad (k=1,\dots,n) \tag{7}$$

with coefficients periodic in $t$, to the system

$$\begin{aligned} \dot{u}_{k,s+1} &= i\mu_k u_{k,s+1} + iu_{k,s+1}[(\lambda_k - \mu_k) + H_{ks}(z_{j,s+1}) + G_{ks}(z_{j,s+1})] \\ &\quad + \Phi_{k,s+1}(u_{j,s+1}, v_{j,s+1}, t), \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{v}_{k,s+1} &= -i\mu_k v_{k,s+1} - iv_{k,s+1}[(\lambda_k - \mu_k) + H_{ks}(z_{j,s+1}) + G_{ks}(z_{j,s+1})] \\ &\quad + \overline{\Phi}_{k,s+1}(u_{j,s+1}, v_{j,s+1}, t) \qquad (k,j=1,\dots,n). \end{aligned}$$

Here $z_{j,s+1} = u_{j,s+1} v_{j,s+1}$,

$$G_{ks}(z_{j,s+1}) = \sum_{\sigma=N_s}^{2N_s-1} a_{k,s+1}^{(\sigma_1,\dots,\sigma_n)} z_{1,s+1}^{\sigma_1} \cdots z_{n,s+1}^{\sigma_n},$$

$$a_{k,s+1}^{(\sigma_j)} = \frac{1}{2\pi i} \int_0^{2\pi} M_{ks}^{(\sigma_j+\delta_{kj},\sigma_j)}(t)\, dt \qquad (k,j=1,\dots,n),$$

and $\Phi_{k,s+1}$ are functions of the same character as $\Phi_{ks}$, containing no terms of degree lower than $(2N_{s+1}+1)$. We note that, by virtue of condition A, $G_{ks}$

are real functions. We define the functions $\varphi_{1s}, \dots, \varphi_{ns}$ as the solution of the following system of differential equations

$$\frac{\partial \varphi_{ks}}{\partial t} + i \sum_{j=1}^{n} \left( \frac{\partial \varphi_{ks}}{\partial u_{js}} u_{js} - \frac{\partial \varphi_{ks}}{\partial v_{js}} v_{js} \right) \{\mu_j + [(\lambda_j - \mu_j) + H_{js}]\}$$
$$- i\varphi_{ks}\{\mu_k + [(\lambda_k - \mu_k) + H_{ks}]\} = iu_{ks}G_{ks} - \sum_{m=2N_s+1}^{4N_s} M_{ks}^{(k_1,\dots,l_n)} u_{1s}^{k_1} \cdots v_{ns}^{l_n}$$
$$+ iu_{ks} \sum_{j=1}^{n} \left( \frac{\partial H_{ks}}{\partial u_{js}} \varphi_{js} + \frac{\partial H_{ks}}{\partial v_{js}} \overline{\varphi}_{js} \right) \qquad (k = 1, \dots, n). \tag{9}$$

Equating in (9) the coefficients of equal powers of $u_{js}, v_{js}$, we obtain first-order linear differential equations, from which $L_{ks}^{(k_j,l_j)}$ are determined as periodic solutions.

If $k_j \neq l_j + \delta_{kj}$ for at least one $j$, then

$$L_{ks}^{(k_1,\dots,l_n)} = -\frac{e^{-i\omega_{ks}t}}{e^{2\pi i\omega_{ks}} - 1} \int_t^{t+2\pi} e^{i\omega_{ks}\tau} M_{ks}^{(k_1,\dots,l_n)}(\tau)\, d\tau$$
$$- \frac{ie^{-i\omega_{ks}t}}{(e^{2\pi i\omega_{ks}} - 1)^2} \sum_{j=1}^{n} \int_t^{t+2\pi} \left\{ \frac{\partial H_{ks}}{\partial z_{js}} \int_\tau^{\tau+2\pi} e^{i\omega_{ks}\xi} \left[ M_{ks}^{(k_1,\dots,k_{j-1},\dots,l_j-1,\dots,l_n)}(\xi) \right.\right.$$
$$\left.\left. + \overline{M}_{ks}^{(l_1,\dots,k_{j-1},\dots,k_j-1,\dots,k_n)}(\xi) \right] d\xi \right\} d\tau, \tag{10}$$

where

$$\omega_{ks} = \sum_{j=1}^{n} \{(k_j - l_j - \delta_{kj})\mu_j + (k_j - l_j - \delta_{kj})[(\lambda_j - \mu_j) + H_{js}]\} \qquad (k = 1, \dots, n). \tag{11}$$

If, however, $k_j = l_j + \delta_{kj}$ for all $j = 1, \dots, n$, then $L_{ks}^{(k_j,l_j)}$ is found as an antiderivative of a certain periodic function with mean value equal to zero. It is easy to see that, for small $|\omega_{ks} - p|$,

$$|e^{2\pi i\omega_{ks}} - 1| > \alpha|\omega_{ks} - p|, \qquad \alpha > 0, \quad p \text{ an integer.}$$

Hence, from (5) and (11) it follows that, if $\Delta_s$ in (7) is sufficiently small, then in (10) the denominator does not vanish. Then system (8) will satisfy all the assumptions made with respect to system (6), if $s$ is replaced by $s+1$, with the exception of condition A.

**Theorem 1.** *Suppose that condition A is satisfied for all $s$. Then, for any $\varepsilon > 0$, in the $\varepsilon$-neighborhood $V_\varepsilon$ of the origin in the phase space of system (2) there exist analytic $n$-dimensional invariant torus-like surfaces, periodic in $t$, $T_{\mu_1,\dots,\mu_n}$, on which there are defined transformations, analytic and with coefficients periodic in $t$, that reduce system (2) to the form*

$$\dot{\xi}_k = i\mu_k\xi_k, \qquad \dot{\eta}_k = -i\mu_k\eta_k \qquad (k = 1, \dots, n),$$

*where $\mu_1, \dots, \mu_n$ satisfy condition (5).*

*Moreover, for any $\chi > 0$ one can specify an $\varepsilon_0$ such that, for all $\varepsilon < \varepsilon_0$, the measure of the set $v_\varepsilon$ of initial points at $t = 0$ of solutions of system (2) that do not belong to $T_{\mu_1,\dots,\mu_n}$ satisfies the inequality*

$$\operatorname{mes} v_\varepsilon < \chi \operatorname{mes} V_\varepsilon.$$

2°. Considering autonomous systems of two equations, A. M. Lyapunov noted [5] two classes of systems for which the equilibrium state is a center—canonical systems and systems that do not change when $t$ is replaced by $-t$ and $y$ by $-y$. Let us pose a more general problem: to investigate the neighborhood of an equilibrium state for analogous systems of order $2n$ with periodic dependence on time.

Consider system (2), and to the assumptions made with respect to it add the following: system (2) does not change under the replacement of $t$ by $-t$ and $y_j$ by $-y_j$ $(j = 1, \dots, n)$. Then one can prove that, for all $s$,

$$M_{ks}^{(k_j,l_j)}(t) = -\overline{M}_{ks}^{(k_j,l_j)}(-t) \qquad (k, j = 1, \dots, n).$$

It follows from this that the mean values of the coefficients $M_{ks}^{(k_j,l_j)}(t)$ are purely imaginary quantities. Consequently, condition A is satisfied; hence, by Theorem 1, we obtain that the equilibrium state of the system under consideration is an analogue of a center in the sense that, for most initial data from a small neighborhood of the origin, the solutions belong to invariant closed surfaces. For stationary systems this assertion was proved in [1]. For canonical systems an analogous result was obtained by V. I. Arnold [6,3] by means of the same Newton method.* If $n = 1$, then the zero solution turns out to be stable in the sense of Lyapunov. In particular, the following holds.

**Theorem 2.** *The zero solution of the differential equation*

$$\ddot{x} + \lambda^2 x = f(x, \dot{x}, t), \tag{12}$$

*where $f(x, \dot{x}, t)$ is an analytic function of $x, \dot{x}$, continuous and $2\pi$-periodic in $t$, containing no linear terms in its expansion, is stable in the sense of Lyapunov if $\lambda$ is irrational, condition (4) is satisfied, and (12) does not change under the replacement of $t$ by $-t$, i.e. $f(x, -\dot{x}, -t) = f(x, \dot{x}, t)$.*

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* Under other assumptions, the question of the applicability of the Newton method to "reversible" systems was also considered by J. Moser [7].

*Note: Figure translations are in progress. See original paper for figures.*