ON THE BRANCHING OF PERIODIC SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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Abstract

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MATHEMATICS

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ON THE BRANCHING OF PERIODIC SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS

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1. The method developed by the authors in papers ($^{1-3}$) is applied here to solve the problem of continuation and branching of periodic solutions of quasilinear differential-difference equations, and new facts are established. In the analytic case we study the problem in a general form and derive for it a branching equation, which is investigated both in the one-dimensional and in the multidimensional cases.

2. Consider the equation

$$dx(t)/dt = Ax(t) + Bx(t-\tau) + \lambda F(t, x(t), x(t-\tau), \lambda), \qquad (1)$$

where $\tau < 2\pi$ is a positive number; A and B are constant real square matrices of order n; λ is a small complex (or real) parameter; $F(t, u, v, \lambda)$ is a continuous vector function, 2π-periodic in t, analytic in u, v, λ in some domain $G = U \times U \times \Lambda$ of their variation, containing the point $u = v = 0, \lambda = 0$. It is assumed that the components of the vector $F = (F_1, \ldots, F_n)$, as functions of t, are expandable in uniformly convergent Fourier series.

Let $\eta(t) \subset U$ be a continuous 2π-periodic solution of the generating equation

$$dy(t)/dt = Ay(t) + By(t-\tau). \qquad (2)$$

We pose the problem of finding all 2π-periodic solutions $x(t, \lambda)$ of equation (1), continuous in (t, λ), satisfying the condition $x(t, 0) = \eta(t)$.

Let the characteristic equation

$$\det\left(A + B\exp(-\tau\mu) - E\mu\right) = 0$$

have m $(m \leq n)$ resonance roots, let these be simple, and let the corresponding m linearly independent 2π-periodic solutions $\varphi_i(t)$ of the generating equation be such that

$$\operatorname{rang}\{\varphi_1(0), \dots, \varphi_m(0)\} = m.$$

If $m > 0$, then

$$\eta(t) = M_1\varphi_1(t) + \dots + M_m\varphi_m(t).$$

It is assumed that the coefficients M_i satisfy the necessary condition (^{4}) for continuation in λ of the solution $\eta(t)$. Without loss of generality one may suppose that $\eta(t) \equiv 0$.

3. Consider the auxiliary equation (cf. (^{4}))

$$dx(t)/dt = Ax(t) + Bx(t-\tau) + \lambda F(t, x(t), x(t-\tau), \lambda) + \lambda \sum_{i=1}^{m} w_i\varphi_i(t), \quad (3)$$

where

$$w_i = \sum_{k=0}^{\infty} \lambda^k w_i^{(k)},$$

and we shall seek continuous 2π-periodic solutions $x(t, a_1, \dots, a_m, \lambda)$ of this equation satisfying the condition

$$x(0, a_1, \dots, a_m, \lambda) = \varphi_1(0)a_1 + \dots + \varphi_m(0)a_m. \quad (4)$$

We shall assume that the vector $\alpha = (a_1, \dots, a_m)$ is such that the right-hand side of (4) belongs to U. Let

$$d_{\mu\nu} = \int_0^{2\pi} (\varphi_\nu(t), \psi_\mu(t))\, dt,$$

where the vectors $\psi_1(t), \dots, \psi_m(t)$ are linearly independent 2π-periodic solutions of the equation adjoint to (2).

Lemma 1. For the 2π-periodic solutions of problem (3)–(4) there is an analogue of Poincaré's theorem; i.e., for sufficiently small a and λ the problem has a unique solution in the class of continuous functions, and it is representable in the form of a convergent series

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots, \quad (5)$$

where $x_k = x_k(t, \alpha, w)$ are analytic functions of α and w, and the vector $w = (w_1, \dots, w_m)$ satisfies the equation

$$w = -d^{-1}\left\{\int_0^{2\pi}(F(t,x(t,\ldots),x(t-\tau,\ldots),\lambda),\psi_i(t))\,dt\right\}_1^m, \qquad d=(d_{\mu\nu}),$$

with a contraction operator (with respect to w) on the right-hand side. Of course, $w_i = w_i(\alpha,\lambda)$ are functions analytic at zero.

4. Consider the equation

$$w_i(\alpha,\lambda)=0 \qquad (i=1,\ldots,m). \tag{6}$$

Let $a = a(\lambda)$ be a small solution of (1) of equation (6). Substituting it into (5), we obtain a solution of the posed problem for equation (1).

Lemma 2. There exists a one-to-one correspondence between the small solutions of equation (6) and the solutions of the posed problem for equation (1).

Let $D = \partial w/\partial a$ at the point $\alpha = 0, \lambda = 0$, and let $\rho = \operatorname{rang} D$. If $\rho = m$, then the solution $\eta(t) \equiv 0$ has a unique continuation in λ, and it is analytic. If $\rho < m$, then, after eliminating ρ components of the vector α from system (6), we obtain the system

$$\Phi_i(\xi_1,\ldots,\xi_r,\lambda)=0, \qquad i=1,\ldots,r; \quad r=m-\rho, \tag{7}$$

where $\xi_1,\ldots,\xi_r$ are the remaining components of the vector α. The functions Φ_i have the following properties: 1. They are analytic at zero. 2. The series Φ_i contain no constant terms and no linear terms in $\xi_1,\ldots,\xi_r$. By virtue of Lemma 2, there exists a one-to-one correspondence between all small solutions $\xi_i = \xi_i(\lambda)$ of system (7) and all small solutions of the posed problem for equation (1). In view of this, system (7) is the bifurcation equation of the problem under study.

In what follows we shall assume that, in system (7), reductions by the highest permissible powers of λ have been carried out and that after this all Φ_i vanish at zero.

System (7) is studied by the methods of paper (^{1}), so that, for the problem under study, both the assertions of Theorems 1-5 of paper (^{2}) and the general considerations of item 5 of paper (^{2}) are valid.

5. To study real solutions of the real equation (1), we restrict ourselves to the case $n=2,\ m=2$. In the one-dimensional case the bifurcation equation has the form

$$\sum_{i=2}^{\infty} L_{i0}\xi^i + \sum_{i=0}^{\infty}\xi^i\sum_{k=1}^{\infty}L_{ik}\lambda^k = 0, \tag{8}$$

where the L_{ik} are real. It is studied with the aid of Newton diagrams; moreover, by virtue of Lemma 2, the number and form of its small real solutions determine the number and form of the real solutions of the posed problem. It is easy, na-

for example, distinguish the cases when the solutions have the form

$$x_i(t,\lambda) = \sum_{k=1}^{\infty} x_{ik}(t)\lambda^{k/2}, \qquad i = 1, 2; \tag{9}$$

$$x_i(t,\lambda) = \sum_{k=1}^{\infty} x_{ik}(t)\lambda^{k}, \qquad i = 1, 2, \tag{10}$$

in some neighborhood of the point $\lambda = 0$. In the two-dimensional case equations (6) and (7) coincide (for $m = n = 2$), and after a nonsingular linear transformation $(a_1, a_2) \to (\xi_1, \xi_2)$ we have

$$\sum_{m_1+m_2\geq s_i} L^{(i)}_{m_1 m_{20}} \xi_1^{m_1} \xi_2^{m_2} + \sum_{m_1+m_2\geq 0} \xi_1^{m_1} \xi_2^{m_2} \sum_{n\geq 1} L^{(i)}_{m_1 m_2 n} \lambda^n = 0, \qquad i = 1, 2, \tag{11}$$

where $L^{(i)}$ are real, $s_i = \operatorname{ord} w_i(a_1, a_2, 0)$, $L^{(i)}_{s_i 00} = 1$. Small solutions of this system are found according to the scheme indicated in (5).

Let us reduce this system to normal form, and then form the resultant

$$R(\xi_2, \lambda) \equiv \sum_{i\geq p} B_{i0}\xi_2^i + \sum_{i\geq 0} \xi_2^i \sum_{k\geq 1} B_{ik}\lambda^k.$$

Investigation of the resultant leads to various conclusions about the number and form of all real solutions of the problem posed. In particular, the following propositions hold.

Theorem 1. Suppose $s_1 = s_2 = 2$, $p = 4$, $q = \operatorname{ord} R(0, \lambda) = 2$, and $B_{21}^2 - 4B_{40}B_{02} \neq 0$. Then the problem posed has as many real solutions, defined for $\lambda \geq 0$ ($\lambda \leq 0$), as the polynomial $B_{40}z^4 + B_{21}z^2 + B_{02}$ ($B_{40}z^4 - B_{21}z^2 + B_{02}$) has real roots, and these solutions are representable in the form (9).

Remark. If, under the conditions of Theorem 1, $B_{21} = 0$, $B_{40} < 0$, then the problem has 4 solutions of the form (9), of which 2 are defined for $\lambda \geq 0$, and 2 for $\lambda \leq 0$.

Theorem 2. Suppose $s_1 = s_2 = 2$, $p = 4$, $q = 3$, $L^{(1)}_{110} \neq L^{(2)}_{110}$, and $B_{12}^2 - 4B_{21}B_{03} \neq 0$. Then the number of real solutions of the problem posed coincides with the number of real roots of the polynomials $f_1(z) = B_{21}z^2 + B_{12}z + B_{03}$, $f_2(z) = B_{40}z^2 + B_{21}$, $f_3(z) = B_{40}z^2 - B_{21}$. Here the roots of $f_1(z)$ correspond to solutions of the form (10), defined in some neighborhood of $\lambda = 0$; the roots of $f_2(z)$ $(f_3(z))$ correspond to solutions of the form (9), defined for $\lambda \geq 0$ $(\lambda \leq 0)$.

Theorem 3. Suppose $s_1 = s_2 = 2,\ p = 4,\ q = 3,\ L^{(1)}_{110} = L^{(2)}_{110}$, and $B_{40}B_{03} < 0$ $(B_{40}B_{03} > 0)$. Then the problem posed has only two real solutions; they are defined for $\lambda \geq 0$ $(\lambda \leq 0)$ and have the form

$$\sum_{k=3}^{\infty} x_{ik}(t)\lambda^{k/4}.$$

As an example, consider the system

$$dx(t)/dt - y(t) = \lambda\left[x^2(t)\cos t + x^2(t-\pi)\cos t - 12\lambda\cos^3 t + \lambda^2 F_1(t, x(t), x(t-\pi), y(t), y(t-\pi), \lambda)\right],$$

$$dy(t)/dt + 2x(t) + x(t-\pi) = \lambda\left[(1+\sin t)x^2(t) - \frac{\xi}{4}x^2(t-\pi) + \lambda^2 F_2(t, x(t), x(t-\pi), y(t)y(t-\pi), \lambda)\right].$$

The branching equation (8) takes the form

$$\xi_1^2 + \frac{7}{6}\xi_1\xi_2 + \frac{7}{12}\xi_2^2 - 3\lambda + \ldots = 0,$$

$$\xi_1^2 + \xi_1\xi_2 + 0\xi_2^2 + 0\lambda + \ldots = 0.$$

Hence we find that $p = 4,\ q = 2,\ B_{40} = 35/144,\ B_{02} = 9,\ B_{21} = -3$, so that, according to Theorem 1, the problem posed for this example has 4 real solutions of the form (9), defined for $\lambda \geq 0$. Using the coefficients of the series F_1, F_2 and the information obtained about the number and form of the solutions, we can easily construct them by the method of undetermined coefficients.

6. Consider the real equation

$$dx(t)/dt = Ax(t) + Bx(t-\tau) + \lambda F(x(t), x(t-\tau), \lambda) \tag{12}$$

under the assumption that A, B, and F satisfy the conditions of item 2, and moreover that the characteristic equation has at least one pair of purely imaginary resonant roots, to which there correspond 2π-periodic solutions $\varphi_{m-1}(t)$ and $\varphi_m(t)$ of equation (2). Then (see (6)) as the generating solution one may take (with a suitable choice of the origin of counting)

$$\eta(t) = M_1\varphi_1(t) + \cdots + M_{m-1}\varphi_{m-1}(t)$$

($M_1, \dots, M_{m-1}$ are constants). The problem is posed of finding all real continuous solutions of equation (12) of period $T(\lambda) = 2\pi(1 + \lambda h_0 + \lambda h(\lambda))$, $h(0) = 0$, which turn into $\eta(t)$ for $\lambda = 0$, and also the function $h(\lambda)$.

As in the nonautonomous case, we compose an auxiliary equation of the form (3) and shall seek its $T(\lambda)$-periodic solutions satisfying the condition

$$x(0, a_1, \dots, a_{m-1}, \lambda) = (M_1 + a_1)\varphi_1(0) + \dots + (M_{m-1} + a_{m-1})\varphi_{m-1}(0). \quad (13)$$

Putting ⁽⁶⁾ $t = \theta(1 + \lambda h_0 + \lambda h(\lambda))$, we reduce the latter initial-value problem to the finding of 2π-periodic solutions of problem (13)–(3), where in (3) the delay τ is replaced by the delay $\tau(1 + \lambda h_0 + \lambda h)^{-1}$. In this way one obtains a system of the form (6)

$$w_i(M_1 + a_1, \dots, M_{m-1} + a_{m-1}, h_0 + h, \lambda) = 0, \qquad i = 1, \dots, m. \quad (14)$$

The necessary condition for continuability in λ of the solution $\eta(t)$ leads to the determination of $M_1, \dots, M_{m-1}, h_0$ from the system $w_i(M_1, \dots, M_{m-1}, h_0, 0) = 0$, $i = 1, \dots, m$, and with such a choice of them the system (14) plays for the given problem the same role as the system (6) in the problem of item 2. Introducing the Jacobi matrix for the system (14) and taking its rank into account, we arrive at the branching equation (7), in which $\xi_1, \dots, \xi_r$ are the remaining components of the vector $\alpha = (a_1, \dots, a_{m-1}, h)$. Thus, the problem is reduced to finding small real solutions of the branching equation.

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Note: Figure translations are in progress. See original paper for figures.

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