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Abstract

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MECHANICS OF CONTINUA

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ASYMMETRIC MECHANICS OF CONTINUA AND AN AVERAGED DESCRIPTION OF TURBULENT FLOWS

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1. In theories of turbulence of liquid and gas flows, various methods of averaging are used, but at the same time assumptions about their equivalence are employed. It is easy to see, however, that in anisotropic turbulent flows, averaging over planes of tensor quantities (stresses, momentum fluxes, etc.) may lead to different results depending on the orientation of the averaging plane relative to the characteristic axes of anisotropy of the turbulently perturbed fluid element. In this case the Reynolds-stress tensor r_{ij} is, in general, asymmetric. The possibility of introducing asymmetric tensors is essential, since in anisotropic cases the presence, on the average, of an orientation of turbulent vortices leads to the appearance of an additional averaged angular momentum. To analyze such a situation it is necessary to use an independent conservation law for angular momentum (see, for example, ([1,2])).

We shall regard a turbulent fluid as a special continuum, whose elementary micromotions are described by a system of Navier–Stokes equations (microcoordinates $x_i,\ i = 1, 2, 3$) under a random choice of initial and boundary conditions. In constructing differential macroequations for such a medium (Reynolds equations), we shall put forward the condition, usual for continuum mechanics, that it is possible to choose such an elementary macrovolume $\Delta V = \Delta X_1 \Delta X_2 \Delta X_3$, whose dimensions are much greater than the internal scales of the microstructure, but much smaller than the characteristic scales of the flows under study. Then the choice of the corresponding averaging rules is determined in passing (by integration over ΔV) from the microequations (Navier–Stokes) to the macroequations (Reynolds).

2. According to asymmetric mechanics ([1,2]), for a complete description of the motion of continua it is necessary to consider not only the equations of conservation of mass and momentum

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0, \qquad \mathbf{J}_\rho = \rho\mathbf{u}; \tag{1}$$

$$\partial \mathbf{P}/\partial t + \nabla \cdot (\mathbf{uP}) = \mathbf{F} + \nabla \mathbf{t}, \qquad \mathbf{P} = \rho \mathbf{u}, \qquad \mathbf{J}_P = \mathbf{uP}, \tag{2}$$

but also the equation of conservation of angular momentum

$$\partial \mathbf{L}/\partial t + \nabla \cdot (\mathbf{uL}) = \mathbf{R} \times \mathbf{F} + \mathbf{G} - \nabla \cdot (\mathbf{t} \times \mathbf{R}) + \nabla \cdot \mathbf{c}, \qquad \mathbf{uL} = \mathbf{J}_L \tag{3}$$

with respect to some point of space (for example, with respect to the origin of coordinates).

Here ρ is the density of the fluid; $\mathbf{J}_\rho$ is the mass flux; $\mathbf{P}$ is the momentum; $\mathbf{J}_P$ is the momentum flux; $\mathbf{F}$ is the density of body forces; $\mathbf{t}$ is the stress tensor; $\mathbf{L} = \mathbf{M} + \mathbf{R} \times \mathbf{P}$ is the total angular momentum; $\mathbf{M}$ is the intrinsic angular momentum for the volume considered; $\mathbf{R}$ is the radius vector; $\mathbf{J}_L$ is the flux of angular momentum $\mathbf{L}$; $\mathbf{G}$ is the density of body couples; $\mathbf{c}$ is the couple-stress tensor.

Subtracting from equation (3) the equation of conservation of the momentum of translational motion of the continuum microparticles

$$\frac{\partial}{\partial t}(\mathbf{R} \times \mathbf{P}) + \nabla \cdot (\mathbf{uR} \times \mathbf{P}) = \mathbf{R} \times \mathbf{F} - \nabla \cdot (\mathbf{t} \times \mathbf{R}) - \mathbf{t}^a$$

leads to the equation of conservation of internal angular momentum

$$\frac{\partial \mathbf{M}}{\partial t} + \nabla \cdot (\mathbf{uM}) = \mathbf{G} + \mathbf{t}^a + \nabla \cdot \mathbf{c}. \tag{4}$$

Here $\mathbf{t}^a$ is the antisymmetric part of the stress tensor $\mathbf{t}$, and the internal moment may be represented in the form $\mathbf{M} = \mathbf{I} \cdot \Phi$, where Φ is the averaged angular velocity of rotation of the microparticles in the differential volume of the medium under consideration.

In the case of an ordinary incompressible viscous fluid having no internal structure, we have

$$t_{ij} = -p\delta_{ij} + \tau_{ij}, \qquad \tau_{ij} = \rho\nu(\partial u_i/\partial x_j + \partial u_j/\partial x_i),$$

where p is the pressure; τ_{ij} are symmetric viscous stresses; ν is the kinematic viscosity; δ_{ij} is the unit tensor. Equation (4) must then reduce to the equation of vortex diffusion obtained by applying the operation ${}^1/_2$ rot to equation (2),

$$\frac{\partial \Phi}{\partial t} + \nabla \cdot (\mathbf{u}\Phi) = \Phi \cdot \nabla \mathbf{u} + \frac{1}{2\rho} \operatorname{rot} \mathbf{F} + \nu \nabla^2 \Phi, \tag{5}$$

where $\Phi = {}^1/_2 \operatorname{rot} \mathbf{u}$, and the equality $\mathbf{M} = \mathbf{I} \cdot \Phi$ may now be regarded as the definition of the moment of inertia $\mathbf{I}$. Comparison of equations (4) and (5) leads to the condition

$$\Phi \cdot \left(\frac{\partial \mathbf{I}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{I} \right) = -\mathbf{I} \cdot (\Phi \cdot \nabla \mathbf{u}) - \mathbf{I} \cdot \frac{1}{2\rho} \operatorname{rot} \mathbf{F} - \mathbf{I} \cdot \nu \nabla^2 \Phi + \mathbf{G} + \nabla \cdot \mathbf{c}. \quad (6)$$

The interpretation of the first term on the right-hand side of condition (5) as the cause of change in the effective moment of inertia of each element is given in Batchelor' s book (^{3}). The introduction of the corresponding additional condition

$$\mathbf{G} + \nabla \cdot \mathbf{c} - \mathbf{I} \cdot \frac{1}{2\rho} \operatorname{rot} \mathbf{F} - \mathbf{I} \cdot \nu \nabla^2 \Phi = 0 \quad (7)$$

closes the system of equations determining laminar flows of a viscous fluid.

3. We shall assume that equations (1)—(3), in the particular case of an incompressible viscous fluid, describe micromotions ($\nabla_i = \partial/\partial x_i$, $\operatorname{rot}_i = \partial/\partial x_j - \partial/\partial x_k$, $i \neq j \neq k$) in the mass of a turbulized viscous fluid. Let us integrate equations (1)—(3) over the volume $\Delta V = \Delta X_1 \Delta X_2 \Delta X_3$, where $\Delta X_i \gg \Delta x_j$. Then the averaged macroequations of conservation will have the form (ε_{ijk} is the antisymmetric Levi-Civita tensor)

$$\frac{\partial \langle \rho \rangle}{\partial t} + \frac{\partial}{\partial X_j} \langle \rho u_j \rangle_j = 0; \quad (8)$$

$$\frac{\partial \langle \rho u_i \rangle}{\partial t} + \frac{\partial}{\partial X_j} \langle \rho u_i u_j \rangle_j = \langle F_i \rangle + \frac{\partial \langle t_{ij} \rangle_j}{\partial X_j}; \quad (9)$$

$$\frac{\partial \langle L_i \rangle}{\partial t} + \frac{\partial \langle L_i u_j \rangle_j}{\partial X_j} = \langle \varepsilon_{ijk} R_{jF} k \rangle + \langle G_i \rangle - \frac{\partial \langle \varepsilon_{ijk} R_k t_{ij} \rangle_j}{\partial X_j} + \frac{\partial \langle c_{ij} \rangle_j}{\partial X_j}. \quad (10)$$

Here $\langle f_i \rangle$ denotes averaging over volume; $\langle \varphi_{ij} \rangle_j$, over the area $\Delta X_i \Delta X_k$, $i \neq j \neq k$, of the functions f_i and φ_{ij}. We introduce the mean velocity U_i according to the rule $\langle \rho \rangle U_i = \langle \rho u_i \rangle$. For an incompressible fluid $\langle \rho \rangle = \rho = \text{const}$. Taking-

taking into account that the field of vectors (velocities) satisfies the condition $\langle u_i \rangle = \langle u_i \rangle_i = \langle u_i \rangle_j = U_i$, we obtain

$$\langle \rho u_i u_j \rangle_j = -r_{ij} + \rho U_{iU} j, \qquad r_{ij} = -\langle \rho v_i v_j \rangle_j, \quad (11)$$

where $v_i = u_i - U_i$ is the velocity fluctuation; $\langle v_i \rangle_i = \langle v_i \rangle_j = 0$. The tensor r_{ij}, just as $T_{ij} = \langle t_{ij} \rangle_j$, is, generally speaking, asymmetric. Let us emphasize that in his original paper (^{4}) Reynolds distinguished the components r_{ij} and r_{ji}.

In transforming equation (10) we shall take into account that

$$\langle \mathbf{L} \rangle = \langle \mathbf{I} \cdot \ \rangle + \langle \mathbf{R} \times \rho \mathbf{u} \rangle =$$

$$= \mathbf{I}^0 \cdot \Omega + \langle \mathbf{I}^* \cdot \omega^* \rangle + \langle \mathbf{R} \rangle \times \rho \mathbf{U} + \langle \mathbf{R}^* \times \rho \mathbf{v} \rangle,$$

$$\langle \mathbf{R} \times \mathbf{F} \rangle = \langle \mathbf{R} \rangle \times \langle \mathbf{F} \rangle + \langle \mathbf{R}^* \times \mathbf{F}^* \rangle,$$

$$\langle c_{ij} \rangle_j = C_{ij}, \quad \langle \varepsilon_{ijk} R_k t_{lj} \rangle_j = \varepsilon_{ikl} \langle R_k \rangle T_{lj} + \varepsilon_{ikl} \langle R_k^* t_{lj}^* \rangle, \quad \langle R_k \rangle = X_k, \qquad (12)$$

$$\langle L_i u_j \rangle_j = \left\langle \left(I_{il} \Phi_l + \varepsilon_{ikl} R_k^* \rho v_l \right) v_j \right\rangle_j + \langle L_j \rangle U_j,$$

where $\mathbf{R}^* = \mathbf{R} - \langle \mathbf{R} \rangle$, $\omega^* = \Phi - \Omega$, $\mathbf{I}^* = \mathbf{I} - \mathbf{I}^0$, $\mathbf{F}^* = \mathbf{F} - \langle \mathbf{F} \rangle$, $\mathbf{t}^* = \mathbf{t} - \mathbf{T}$ are the corresponding fluctuations, $\Omega = \frac{1}{2} \operatorname{rot} \mathbf{U}$, $\mathbf{I}^0$ is a certain effective moment of inertia, and it is again assumed that $\langle L_i \rangle = \langle L_i \rangle_j$. One may also introduce the effective angular velocity ω of turbulent vortices by the equality $\langle \mathbf{I}^* \cdot \omega^* \rangle = \mathbf{I}^0 \cdot \omega$. Then (with $D/Dt = \partial/\partial t + \mathbf{U} \cdot \nabla$, $\nabla_i = \partial/\partial X_i$) we obtain

$$\frac{D}{Dt} \left\{ \mathbf{I}^0 \cdot (\Omega + \omega) + \langle \mathbf{R} \rangle \times \rho \mathbf{U} + \langle \mathbf{R}^* \times \rho \mathbf{v} \rangle \right\} =$$

$$= \langle \mathbf{R} \rangle \times \langle \mathbf{F} \rangle + \langle \mathbf{G} \rangle + \langle \mathbf{R}^* \times \mathbf{F}^* \rangle + \nabla \cdot (\mathbf{T} \times \langle \mathbf{R} \rangle) + \nabla \cdot \mathbf{C} + \nabla \cdot \mu. \qquad (13)$$

Multiplying (vectorially) equation (9) by $\langle \mathbf{R} \rangle$ and subtracting the product from equation (13), we obtain the conservation equation for the internal moment in a turbulent fluid

$$\frac{D}{Dt} \left\{ \mathbf{I}^0 \cdot (\Omega + \omega) + \langle \mathbf{R}^* \times \rho \mathbf{v} \rangle \right\} = \langle \mathbf{G} \rangle + \langle \mathbf{R}^* \times \mathbf{F}^* \rangle + \nabla \cdot \mathbf{C} + \nabla \cdot \mu + \mathbf{T}^a + \mathbf{r}^a,$$

$$\mu_{ij} = -\langle I_{ik} \Phi_k v_j \rangle_j - \langle \varepsilon_{ikl} R_k^* \rho v_l v_j \rangle_j - \langle \varepsilon_{ijl} R_k^* t_{lj} \rangle. \qquad (14)$$

Here $\mathbf{T}^a$, $\mathbf{r}^a$ are the antisymmetric parts of the tensors of the averaged viscous and Reynolds stresses. The averaged motion is also characterized by additional moment stresses (owing to the fluctuational transport of the momentum moment of macroparticles, and also owing to the inhomogeneity of the fields of fluctuational impulse and viscous stresses) and by an additional mass moment (owing to the nonuniformity of the distribution of mass forces in the volume ΔV).

4. Closing equations for the system of equations of turbulent motion may be constructed according to the principles developed in semiempirical theories of turbulence. The basic assumptions of such theories will be formulated as follows.

First, we shall relate the averaged viscous stresses to the field of mean velocities U_i by the same relations as in a nonturbulent Newtonian fluid,

$$T_{ij} = -P\delta_{ij} + \nu\rho\left(\partial U_j/\partial X_i + \partial U_i/\partial X_j\right), \qquad T_{ij} = T_{ji}. \tag{15}$$

Thus, the antisymmetric part of the tensor of perturbed viscous stresses is neglected: $\mathbf{T}^a = 0$. (Let us note that the condition $\mathbf{T}^a \neq 0$ is essential in the analysis of laminar flows of a fluid with suspended rotating foreign particles (^{5}).)

Let us choose the moment of inertia $\mathbf{I}^0$ so that

$$(\Omega + \omega) \cdot \frac{D\mathbf{I}^0}{Dt} = -\mathbf{I}^0 \cdot \{(\Omega + \omega) \cdot \nabla\mathbf{U}\} \tag{16}$$

Then, under the condition (cf. (6) and (7))

$$\frac{D}{Dt}\langle\mathbf{R}^* \times \rho\mathbf{v}\rangle + \langle\mathbf{G}\rangle - \mathbf{I}^0\frac{1}{2\rho}\,\mathrm{rot}\langle\mathbf{F}\rangle + \nabla\cdot\mathbf{C} + \langle\mathbf{R}^* \times \mathbf{F}^*\rangle - \mathbf{I}\nu\nabla^2(\ \ + \omega) = 0 \tag{17}$$

equation (14) takes the form

$$\mathbf{I}^0 \cdot \frac{D(\ \ + \omega)}{Dt} = \mathbf{I}^0 \cdot \{(\ \ + \omega)\cdot\nabla\mathbf{U}\} + \mathbf{I}^0 \cdot \nu\nabla^2(\ \ + \omega) + \nabla\cdot\mu + \mathbf{r}^a. \tag{18}$$

The second term on the right-hand side of equation (18) corresponds to the viscous dissipation of the mean turbulent vortex.

Secondly, the additional forces and moments appearing because of the turbulization of the liquid will be related by ordinary tensor relations to the field of mean velocities, assuming, however, that the transfer coefficients introduced in this way are functions of the microstructure of the turbulent medium. Then (cf. the constructions for laminar flows in asymmetric hydrodynamics (^{6})) we have

$$r_{ij} - r^a_{ij} = A_{ijkl}\left[\frac{\partial U_k}{\partial X_l} + \frac{\partial U_l}{\partial X_k}\right], \tag{19}$$

$$\mu_{ij} = B_{ijkl}\frac{\partial(\Omega_k + \omega_k)}{\partial X_l}, \tag{20}$$

$$r_{ij}^{a} = D_{ijkl}\varepsilon_{klm}\omega_m. \tag{21}$$

To determine the transfer coefficients A_{ijkl}, B_{ijkl}, D_{ijkl}, it is necessary to introduce hypotheses on the kinetics of mixing in a turbulent flow, making use, for example, of Prandtl' s ideas ([7]) on the turbulent transfer of momentum (to specify relation (19)) and Taylor' s ([8]) on the turbulent transfer of vorticity (to specify relation (20)), as well as the general principles of similarity theory ([9]).

Let us note that subsequent refinements of the averaged description of turbulent flows will require, generally speaking, raising the order of the moments considered for the fields of random quantities and the corresponding introduction, into semiempirical relations, of derivatives of ever higher order.

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Note: Figure translations are in progress. See original paper for figures.

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