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COMPLEXITY OF  
REALIZING  
MONOTONE  
SYMMETRIC  
FUNCTIONS BY  
FORMULAS IN THE  
BASIS  $(\vee, \&, \neg)$**

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**Abstract**

**Full Text**

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**CYBERNETICS AND CONTROL THEORY**

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**ESTIMATES OF THE COMPLEXITY OF REALIZING MONOTONE SYMMETRIC FUNCTIONS BY FORMULAS IN THE BASIS  $\vee, \&, \neg$**

*(Presented by Academician L. V. Kantorovich, January 28, 1969)*

A monotone symmetric function of the algebra of logic with threshold  $p$  is the function  $f_p(x_1, \dots, x_n)$  defined by the formula

$$f_p(x_1, \dots, x_n) = \bigvee_{1 \leq i_1 < i_2 < \dots < i_p \leq n} .$$

Let  $L_{\Pi}(\sigma_n)$  be the smallest number of letters sufficient for realizing any symmetric function of  $n$  arguments by a formula, and let  $L_p(n)$  be the smallest number of letters sufficient for realizing, by a formula,  $f_p(x_1, \dots, x_n)$ .

This note is devoted to finding upper estimates for  $L_p(n)$ . The best upper estimate for  $L_{\Pi}(\sigma_n)$  is the estimate obtained by O. B. Lupanov <sup>(1)</sup>:

$$L_{\Pi}(\sigma_n) \preccurlyeq n^{\log_2 \log_2 n(1+\delta_n)}, \quad \text{where } \delta_n \rightarrow 0, n \rightarrow \infty.$$

If  $p$  is fixed, then the best upper estimate for  $L_p(n)$  is the estimate obtained by the author <sup>(2)</sup>:

$$L_p(n) \preccurlyeq n \frac{(\log_2 p)^{p-1}}{(\log_2 \log_2 n)^{p-2}} *,$$

and the best lower estimate is the estimate obtained by R. E. Krichevskii <sup>(3)</sup>

$$L_2(n) \succcurlyeq n \log_2 n. \tag{1}$$

By a formula of depth 3 we shall mean a disjunction of conjunctions of disjunctions of variables.

In the present note the following results are obtained:

1. For fixed  $p$ ,  $L_p(n) \asymp n \log_2 n$ , and the estimate is attained even on formulas of depth 3. More precisely, the following inequality holds:

2.

$$L_p^3(n) \preceq n \log_2 n \cdot e^{p(1+\alpha_p)},$$

where  $\alpha_p \rightarrow 0$ ,  $p \rightarrow \infty$ , and  $L_p^3(n)$  is the complexity of a minimal formula of depth 3 realizing  $f_p(x_1, \dots, x_n)$ .

3.

$$L_p(n) \preceq 2^{1/4(\log_2 p)^2(1+\beta_p)} n \log_2 n,$$

where  $\beta_p \rightarrow 0$ ,  $p \rightarrow \infty$ .

4.

$$L(n, p) \preceq 2^{1/4(\log_2 p)^2(1+\gamma_p)} n \log_2 n,$$

where  $\gamma_p \rightarrow 0$ ,  $p \rightarrow \infty$ , and  $L(n, p)$  is the smallest number of letters sufficient for realizing, by a formula in the basis  $\vee, \&, \neg$ , a symmetric function whose working numbers lie between 0 and  $p$ . It should be noted that all estimates are obtained ineffectively, i.e., the formulas on which these estimates are attained are not constructed.

**Theorem 1.** For  $L_p(n)$  the inequality

$$L_p(n) \preceq 2^{1/4(\log_2 p)^2(1+\beta_p)} n \log_2 n, \quad \text{where } \beta_p \rightarrow 0, p \rightarrow \infty,$$

holds.

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\* In this note the following notation is used:  $a(m) \preceq b(m)$  means that there exist constants  $c_1$  and  $c_2$  such that for  $m > c_1$  the inequality  $a(m) \leq c_2 b(m)$  holds;  $a(m) \succcurlyeq b(m)$ ,  $-b(m) \preceq a(m)$ ;  $a(m) \asymp b(m)$ ,  $-a(m) \preceq b(m)$  and  $a(m) \succcurlyeq b(m)$ .

**Proof.** Let first  $n = 2^m$ ,  $p = 2^k$  ( $1 \leq k \leq m$ ). Define the class of formulas  $M(n, l_0, l_1, \dots, l_{k-1})$  and choose the parameters  $l_0, \dots, l_{k-1}$  so that in this class there exists a formula realizing  $j_p(x_1, \dots, x_n)$ . Consider a tree  $S$  with root (4). Let the number of levels of the tree be  $2k+2$ , and let the number of arcs issuing from an arbitrary vertex of the  $i$ -th level be equal to 2 if  $i$  is odd and  $i \neq 2k+1$ , equal to  $n/p$  if  $i = 2k$ , and equal to  $l_j$  if  $i = 2j$  ( $0 \leq j \leq k-1$ ). To each path issuing from a vertex of the  $i$ -th level we assign the symbol  $\vee$ , if  $i$  is odd, and  $\&$ , if  $i$  is even and  $i \neq 2k$ . To the root we assign the symbol  $\vee$ . For example, Fig. 1 shows the tree for  $n = 2^1$  and  $p = 2^1$ ,  $l_0 = 2$ . Let  $\mathfrak{N} = \{x_1, \dots, x_n\}$  be the set of  $n$  binary variables. Put  $n_j = n/2^j$  ( $0 \leq j \leq k$ ). To each terminal vertex we assign a variable from  $\mathfrak{N}$  in such a way that condition A is satisfied.

**Condition A.** Let  $a$  be an arbitrary vertex of the  $(2j-1)$ -st level,  $1 \leq j \leq k$ . Then the sets of variables assigned to the terminal vertices of the subtrees generated by the vertices  $a_1$  and  $a_2$  of the  $2j$ -th level following  $a$  must not intersect, and each of them must contain  $n_j$  elements.

Fig. 1

Figure 1: Fig. 1

Let variables be assigned in some way to the terminal vertices of  $S$  so that condition A is satisfied. Consider the subtree  $S(a)$  generated by the vertex  $a$  of the  $2i$ -th level. To this subtree one can uniquely associate a formula. This formula realizes a function equal to the disjunction of some conjunctions of length  $p_i = p/2^i$  in the  $n_i$  variables assigned to the terminal vertices of the subtree  $S(a)$ . Thus, for example, the tree in Fig. 1 corresponds to the formula

**Fig. 1**

$$F(x_1, \dots, x_4) = x_1x_2 \vee x_1x_3 \vee x_1x_4 \vee x_2x_3 \vee x_2x_4 \vee x_3x_4.$$

We shall regard two formulas corresponding to the subtree  $S(a)$  as identical if and only if, for an arbitrary vertex of the  $2k$ -th level belonging to  $S(a)$ , the sets of variables assigned to the terminal vertices of the subtree generated by it coincide.

Denote by  $M(n_i, l_i, \dots, l_{k-1})$  the set of all formulas corresponding to the subtree  $S(a)$  generated by a vertex  $a$  of the  $2i$ -th level, and the number of elements of this set by  $|M(n_i, l_i, \dots, l_{k-1})|$ . Then

$$|M(n_i, l_i, \dots, l_{k-1})| = \left( C_{n_i}^{n_i/2} |M(n_{i+1}, l_{i+1}, \dots, l_{k-1})|^2 \right)^{l_i},$$

$$i = 0, \dots, k-2, \tag{2}$$

$$|M(n_{k-1}, l_{k-1})| = \left( C_{n_{k-1}}^{n_{k-1}/2} \right)^{l_{k-1}}.$$

Let

$$C_{n_i}^{n_i/2} |M(n_{i+1}, l_{i+1}, \dots, l_{k-1})|^2 = R(n_{i+1}, l_{i+1}, \dots, l_{k-1}).$$

Then

$$|M(n_i, l_i, l_{i+1}, \dots, l_{k-1})| = \left( R(n_{i+1}, l_{i+1}, \dots, l_{k-1}) \right)^{l_i}. \tag{3}$$

We shall say that  $F \in M(n_i, l_i, \dots, l_{k-1})$  does not generate the conjunction  $x_{t_1} \cdots x_{t_i}$  (the  $x_{t_j}$  are taken from the set of variables on which  $F$  depends) if the reduced disjunctive normal form of the function realized by  $F$  does not contain this conjunction. Clearly, the number of formulas not generating a concrete

conjunction is the same for all conjunctions. Let  $|\overline{M}(n_i, l_i, \dots, l_{k-1})|$  be the number of such formulas for any conjunction. Then

$$|\overline{M}(n_i, l_i, \dots, l_{k-1})| = (\overline{R}(n_{i+1}, l_{i+1}, \dots, l_{k-1}))^{l_i},$$

where

$$\begin{aligned} \overline{R}(n_{i+1}, l_{i+1}, \dots, l_{k-1}) &= R(n_{i+1}, l_{i+1}, \dots, l_{k-1}) - \\ &- C_{p_i}^{p_i/2} C_{n_i - p_i}^{n_i/2 - p_i/2} (|M(n_{i+1}, l_{i+1}, \dots, l_{k-1})| - |\overline{M}(n_{i+1}, l_{i+1}, \dots, l_{k-1})|)^2, \end{aligned} \quad (4)$$

$$i = 0, \dots, k-2;$$

$$|\overline{M}(n_{k-1}, l_{k-1})| = (C_{n_{k-1}}^{n_{k-1}/2} - 2C_{n_{k-1}-2}^{n_{k-1}/2-1})^{l_{k-1}}.$$

Put

$$\frac{|\overline{M}(n_i, l_i, \dots, l_{k-1})|}{|M(n_i, l_i, \dots, l_{k-1})|} = a_i, \quad i = 0, \dots, k-1.$$

Then, taking into account (2), (3), (4), we have

$$\begin{aligned} a_i &= \left( 1 - \frac{C_{p_i}^{p_i/2} C_{n_i - p_i}^{n_i/2 - p_i/2}}{C_{n_i}^{n_i/2}} (1 - a_{i+1})^2 \right)^{l_i}, \quad i = 0, \dots, k-2. \\ a_{k-1} &= \left( 1 - 2 \frac{C_{n_{k-1}-2}^{n_{k-1}/2-1}}{C_{n_{k-1}}^{n_{k-1}/2}} \right)^{l_{k-1}}. \end{aligned} \quad (5)$$

It is not difficult to verify that if  $l_0, \dots, l_{k-1}$  are chosen from the condition  $a_0 < 1/C_n^p$ , then in the class  $M(n, l_0, \dots, l_{k-1})$  there is a formula realizing the function equal to the disjunction of all conjunctions of length  $p$  in the variables  $x_1, \dots, x_n$ , i.e.  $f_p(x_1, \dots, x_n)$ . Choose  $l_0, \dots, l_{k-1}$  successively from the condition

$$a_i \leq \frac{1}{2}, \quad i = 1, \dots, k-1, \quad a_0 < 1/C_n^p. \quad (6)$$

Since

$$1 - 2C_{n_{k-1}-2}^{n_{k-1}/2-1} / C_{n_{k-1}}^{n_{k-1}/2} \leq \frac{1}{2},$$

then for  $l_{k-1} = 1$  condition (6) is fulfilled for  $i = k - 1$ . Suppose  $l_{i+1}, \dots, l_{k-1}$  have already been chosen so that condition (6) is fulfilled for

$$a_{i+1}, \dots, a_{k-1} \quad (1 \leq i \leq k - 2).$$

Then the inequality is valid

$$\begin{aligned} 1 - \frac{C_{p_i/2}^{p_i/2} C_{n_i-p_i}^{n_i/2-p_i/2}}{C_{n_i}^{n_i/2}} (1 - a_{i+1})^2 &\leq 1 - \frac{1}{4} \frac{C_{p_i/2}^{p_i/2} C_{n_i-p_i}^{n_i/2-p_i/2}}{C_{n_i}^{n_i/2}} \leq \\ &\leq \{ \text{using the refinement of Stirling's formula (5)} \} \leq \\ &\leq 1 - \frac{1}{4} \frac{\sqrt{2/\pi} 2^{p_i} \sqrt{2/\pi} 2^{n_i-p_i} \sqrt{n_i}}{i^{1/3} \sqrt{p_i} \sqrt{n_i - p_i} \sqrt{2/\pi} 2^{n_i}} \leq 1 - \frac{1}{c} \frac{1}{\sqrt{1 - p/n} \sqrt{p_i}}. \end{aligned} \quad (7)$$

Choose  $l_i$  from the condition

$$l_i > 8\sqrt{1 - p/n} \sqrt{p_i}; \quad (8)$$

Then

$$\begin{aligned} a_i &\leq \{ \text{taking into account (7), (8)} \} \leq \left( 1 - \frac{1}{8\sqrt{1 - p/n} \sqrt{p_i}} \right)^{8\sqrt{1 - p/n} \sqrt{p_i}} \leq \\ &\leq \{ \text{since } \log_2(1 - x) \leq -x, 0 \leq x < 1 \} \leq \frac{1}{2}. \end{aligned}$$

Thus, if  $l_i$  ( $1 \leq i \leq k - 2$ ) are chosen in accordance with (8), and  $l_{k-1}$  is set equal to 1, then (6) will be fulfilled for  $i = 1, \dots, k - 1$ . Let  $l_1, \dots, l_{k-1}$  be chosen in the indicated way. Choose  $l_0$  from the condition:

$$l_0 > 8\sqrt{1 - p/n} p^{3/2} \log_2 n. \quad (9)$$

Then, taking into account that (7) is now also valid for  $i = 0$ , and also the inequality  $\log_2 C_n^p < p \log_2 n$ , we have

$$a_0 < 1/C_n^p.$$

Let  $l_0, \dots, l_{k-1}$  be chosen according to (8), (9), and let  $l_{k-1} = 1$ . Then there exists a formula  $F \in M(n, l_0, \dots, l_{k-1})$  that realizes  $f_p(x_1, \dots, x_n)$ . The complexity of this formula satisfies the inequality

$$L_F(n) \leq \frac{n}{p} 2l_0 2l_1 \dots 2l_{k-1} = nl_0 \dots l_{k-1},$$

i.e., there exists a formula whose complexity satisfies the inequality

$$L_F(n) \leq n \prod_{i=1}^{k-1} (8\sqrt{1-p/n}\sqrt{p_i} + 1) (8\sqrt{1-p/n}p^{3/2} \log_2 n + 1) \leq 2^{\frac{1}{4}(\log_2 p)^2(1+\tau_p)} n \log_2 n,$$

where  $\tau_p \rightarrow 0$  as  $p \rightarrow \infty$ .

Thus, we have

$$L_p(n) \leq 2^{\frac{1}{4}(\log_2 p)^2(1+\tau_p)} n \log_2 n, \quad (10)$$

where  $\tau_p \rightarrow 0$ ,  $p \rightarrow \infty$  ( $n = 2^m$ ,  $p = 2^k$ ,  $1 \leq k \leq m-1$ ).

For  $p = n = 2^m$ , inequality (10), obviously, is also valid. Let  $n$  and  $p$  be arbitrary. Then the validity of the asserted theorem is established with the aid of (10) and the relations

$$\begin{aligned} f_p(x_1, \dots, x_n) &= f_q(x_1, \dots, x_n, \underbrace{1, \dots, 1}_{q-p}), \\ f_q(x_1, \dots, x_{n+q-p}) &= f_q(x_1, \dots, x_{n+q-p}, \underbrace{0, \dots, 0}_{N-(n+q-p)}). \end{aligned} \quad (11)$$

Theorem 1 is proved.

**Theorem 2.** The inequality

$$L_p^3(n) \leq l^{p(1+\alpha_p)} n \log_2 n, \quad \text{where } \alpha_p \rightarrow 0, p \rightarrow \infty,$$

is valid.

The proof is analogous to the proof of Theorem 1.

**Corollary 1.** For fixed  $p$ ,

$$L_p(n) \asymp n \log_2 n.$$

The proof is easily obtained using Theorem 2 and relations (1) and (11).

**Corollary 2.**

$$L(n, p) \leq 2^{\frac{1}{4}(\log_2 p)^2(1+\gamma_p)} n \log_2 n, \quad \gamma_p \rightarrow 0, \quad p \rightarrow \infty.$$

The proof follows from Theorem 1 and the relation

$$\varphi_p(x_1, \dots, x_n) = f_p(x_1, \dots, x_n) f_{p+1}(\bar{x}_1, \dots, \bar{x}_n),$$

where  $\varphi_p(x_1, \dots, x_n)$  is the elementary symmetric function with working number  $p$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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