

# MIXED CONTROL IN A DIFFERENTIAL GAME

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**Abstract**

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*MATHEMATICS*

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## MIXED CONTROL IN A DIFFERENTIAL GAME

Let us consider a controlled system described by the equation

$$\dot{x} = f^{(1)}(t, x, u) - f^{(2)}(t, x, v), \quad (1)$$

where  $x$  is an  $n$ -dimensional phase vector;  $u$  and  $v$  are  $r$ -dimensional control vectors belonging to the first and second players, respectively; the realizations  $u[t]$  and  $v[t]$  are constrained by

$$u[t] \in \mathcal{U}, \quad v[t] \in \mathcal{V}, \quad (2)$$

where  $\mathcal{U}$  and  $\mathcal{V}$  are bounded and closed sets; the functions  $f^{(i)}$  are continuous in all arguments, differentiable with respect to  $x$ , and satisfy the inequalities

$$|x' f_x^{(i)}(t, x, w)| \leq \lambda (1 + \|x\|^2) \quad (\lambda = \text{const}; i = 1, 2).$$

(Vectors are treated as column vectors, the symbol  $\|q\|$  denotes the Euclidean norm of the vector  $q$ , and a prime as a superscript denotes transposition.)

The problem of the first player is to bring the phase point  $x[t]$  to a prescribed set  $\mathcal{M}$ ; the second player prevents this. Similar game situations have been considered, for example, in the works <sup>(1-7)</sup>. In the present note we indicate conditions for the successful termination of the game by the first player when his strategy  $U^0$  is formalized within the framework of the theory of differential equations in contingencies <sup>(8,9)</sup>. This strategy, which mixes the controls  $u$ , is based on the rule of extremal aiming generalized here <sup>(7)</sup>, formalized earlier for the differential game of pursuit in <sup>(10,11)</sup>. The strategy  $U^0$  proves to be effective in cases of stable absorption <sup>(12)</sup>, which correspond to solutions of game control problems under conditions of discrimination of the second player <sup>(2-6)</sup>.

Admissible strategies  $U$  and  $V$  are defined as follows. Denote by the symbol  $F^{(i)}(t, x)$  the convex hull of the set  $\{f = f^{(i)}(t, x, w) : w = u \in \mathcal{U} \text{ or } w = v \in \mathcal{V}\}$ . A strategy  $W = U$  ( $W = V$ ) will be specified by a system of vector sets  $F^{(1)}(t, x; U)$  ( $F^{(2)}(t, x; V)$ ), assigned to each possible position  $\{t, x\}$ ; it will be

admissible (on the interval  $t_0 \leq t \leq \vartheta$ ) if the inclusions  $F^{(i)}(t, x; W) \in F^{(i)}(t, x)$  hold and if the sets  $F^{(i)}(t, x; W)$  are convex and upper semicontinuous with respect to inclusion relative to variation of  $t$  and  $x$  for all  $x$  and for almost all  $t \in [t_0, \vartheta]$ . The initial position  $\{t_0, x_0\}$  and a pair of admissible strategies  $\{U, V\}$  generate motions  $x[t]$  ( $t_0 \leq t \leq \vartheta$ ) of system (1), which are absolutely continuous functions satisfying, for almost all  $t$ , the equation  $\dot{x}[t] = f^{(1)}[t] - f^{(2)}[t]$ , where

$$f^{(1)}[t] \in F^{(1)}(t, x[t]; U), \quad f^{(2)}[t] \in F^{(2)}(t, x[t]; V).$$

Let  $\mathcal{M}_0$  be some bounded, convex, and closed set contained in  $\mathcal{M}$ . We shall say that  $\vartheta$  is the moment of absorption of the set  $\mathcal{M}_0$  by the process  $x(t)$  (1) from the position  $\{t_*, x_*\}$  ( $t_* \leq \vartheta$ ), if the following condition is fulfilled:

1°. To each measurable function  $v(t) \in \mathcal{V}(t_* \leq t \leq \vartheta)$  and to any number  $a > 0$  one can assign a measurable function  $u(t) \in \mathcal{U}(t_* \leq t \leq \vartheta)$  such that the pair of controls  $\{u(t), v(t)\}$  transfers the motion  $x(t)$  (1) from the position  $\{t_*, x_*\}$  to the state  $x(\vartheta)$ , whose distance from the set  $\mathcal{M}_0$  does not exceed  $a$ .

Let the symbol  $\mathcal{W}(t_*, \vartheta)$  ( $t_* \leq \vartheta$ ) denote the set of all those  $x$  for which  $\vartheta$  is an absorption time of the set  $\mathcal{M}_0$  by the process  $x(t)$  (1) from the position  $\{t_*, x\}$ . We shall say that, for  $t_0 \leq t \leq \vartheta$ , absorption is stable if the following conditions are satisfied.

2°. The sets  $\mathcal{W}(t, \vartheta)$  are convex.

3°. Whatever  $t_* < \vartheta$ , if  $x_* \in \mathcal{W}(t_*, \vartheta)$ , then for each positive number  $\Delta \leq \Delta_0$ ,  $\Delta \leq \vartheta - t_*$  ( $\Delta_0 > 0$ —const), for every pair  $\{v(t), a\}$  ( $v(t) \in \mathcal{V}$ ,  $t_* \leq t \leq t_* + \Delta$ ,  $a > 0$ ) one can specify a function  $u(t) \in \mathcal{U}(t_* \leq t \leq t_* + \Delta)$  such that the pair of controls  $\{u(t), v(t)\}$  transfers the motion  $x(t)$  (1) from the position  $\{t_*, x_*\}$  to the state  $x(t_* + \Delta)$ , whose distance from the set  $\mathcal{W}(t_* + \Delta, \vartheta)$  does not exceed  $a$ .

Thus, let some set  $\mathcal{M}_0 \subset \mathcal{M}$  be chosen and let the number  $\vartheta$  be fixed. Construct the strategy  $U^0$ . If the position  $\{t, x\}$  ( $t < \vartheta$ ) is such that  $x \in \mathcal{W}(t, \vartheta)$ , then we set  $F^{(1)}(t, x; U^0) = F^{(1)}(t, x)$ . If, however,  $x \notin \mathcal{W}(t, \vartheta)$ , then we construct the smallest closed  $\varepsilon$ -neighborhood  $\mathcal{W}_\varepsilon^0(t, x)(t, \vartheta)$  of the set  $\mathcal{W}(t, \vartheta)$  containing the point  $x$ . Let  $s^0$  be a unit vector of the inward normal to the boundary of the domain  $\mathcal{W}_\varepsilon^0$  at the point  $x$ . Then, as  $F^{(1)}(t, x; U^0)$ , we choose the set of all vectors  $f^0 \in F^{(1)}(t, x)$  satisfying the maximum conditions

$$s^{0'} f^0 = \max_f s^{0'} f \quad \text{for } f \in F^{(1)}(t, x). \quad (3)$$

**Theorem.** *Let  $\vartheta > t_0$  be an absorption time for the initial position  $\{t_0, x_0\}$ . If, on the interval  $t_0 \leq t \leq \vartheta$ , the process  $x(t)$  (1) absorbs the set  $\mathcal{M}_0$  stably, then the admissible strategy  $U^0$  ensures that the point  $x[t]$  is brought onto the set  $\mathcal{M}_0$  no later than at the time  $t = \vartheta$ , whatever the admissible strategy  $V$  and whatever motion  $x[t]$  possible under the chosen strategies  $U^0$  and  $V$  may be.*

The main point of the proof is the verification of the estimate  $\varepsilon^0[t] \leq \varepsilon^0[t_*] \exp \chi(t - t_*)$  ( $\varepsilon^0[t] = \varepsilon^0(t, x[t])$ ,  $\chi = \text{const}$ ,  $t \geq t_*$ ), which is valid under the conditions of stable absorption along any motion  $x[t]$  generated by the strategies  $U^0$  and  $V$ . Let us also note that continuous dependence of the sets  $\mathcal{W}(t, \vartheta)$  on  $t$  is not assumed; however, in our case, as  $t$  increases, jump-like changes of  $\mathcal{W}(t, \vartheta)$  in the direction of contraction are impossible.

An effective formulation of the absorption conditions, the construction of the sets  $\mathcal{W}(t, \vartheta)$  needed for determining the strategy  $U^0$  (3), and the verification of stability of absorption in the general case are difficult. If equation (1) is linear, i.e.,

$$\dot{x} = A(t)x + B(t)u - C(t)v + f(t), \quad (4)$$

then the set  $\mathcal{W}(t, \vartheta)$  can be defined as the totality of all those points  $x$  that satisfy the inequality

$$\begin{aligned} & \rho^{(1)}(t, \vartheta, l) + \rho^0(l) - \rho^{(2)}(t, \vartheta, l) + \\ & + l'X(\vartheta, t)x + l' \int_t^{\vartheta} X(\vartheta, \tau)f(\tau) d\tau \geq 0, \end{aligned} \quad (5)$$

whatever the vector  $l$  may be. Here  $X(t, \tau)$  is the fundamental matrix of solutions of the homogeneous part of equation (4), and the quantities  $\rho^{(i)}$  and  $\rho^0$  are defined

by the equalities

$$\begin{aligned} \rho^{(1)}(t, \vartheta, l) &= \max_{u(\tau) \in \mathcal{U}^*} \left( \int_t^{\vartheta} l'X(\vartheta, \tau)B(\tau)u(\tau) d\tau \right), \\ \rho^{(2)}(t, \vartheta, l) &= \max_{v(\tau) \in \mathcal{V}^*} \left( \int_t^{\vartheta} l'X(\vartheta, \tau)C(\tau)v(\tau) d\tau \right), \\ \rho^0(l) &= \max_q l'q \quad \text{for } -q \in \mathcal{M}_0, \end{aligned}$$

where  $\mathcal{U}^*$  and  $\mathcal{V}^*$  are the convex hulls of the sets  $\mathcal{U}$  and  $\mathcal{V}$  from (2). Condition (5) is equivalent to the absorption conditions in the case of a linear differential pursuit game, where the instant  $\vartheta$  is determined from the condition of absorption of the attainability domain of the pursued object by the attainability domain of the pursuer (7, 10, 11). Namely, in the present case, with the original problem one should associate the problem of pursuing one controlled object

$$\dot{z} = A(t)z + C(t)v$$

by another

$$\dot{y} = A(t)y + B(t)u + f(t),$$

where the goal of pursuit must be the capture of the point  $z[t]$  in a “ $\mathcal{P}$ -neighborhood” of the point  $y[t]$ , according to the condition

$$z[t] - y[t] = p \in \mathcal{P} = -\mathcal{M}_0.$$

For the position  $\{t_*, y_*, z_*\}$  ( $y_* - z_* = x_*$ ), the instant  $\vartheta$  is then determined by the inclusion

$$G^{(2)}(t_*, z_*, \vartheta) \subset G_P^{(1)}(t_*, y_*, \vartheta),$$

where  $G^{(2)}$  is the attainability domain by the instant  $\vartheta$  for the motion  $z(t)$  from the state  $z(t_*) = z_*$  under controls  $v(t) \in \mathcal{V}^*$ , and  $G_P^{(1)}$  is the attainability domain by the instant  $\vartheta$  for the motion  $y(t)$  from the state  $y(t_*) = y_*$  under controls

$$u_p(t) = u(t) + p\delta(t - \vartheta) \quad (u(t) \in \mathcal{U}^*, p \in \mathcal{P}, \delta(t) \text{ is the delta function}).$$

The stability of absorption is then guaranteed in any case if the boundary  $H_{P,\varepsilon^0}$  of the minimal  $\varepsilon$ -neighborhood  $G_{P,\varepsilon^0}^{(1)}$  of the domain  $G_P^{(1)}$ , containing the domain  $G^{(2)}$ , for every  $\varepsilon^0 > 0$  intersects this domain  $G^{(2)}$  in the set  $Q^0$ , which is contained entirely in one hyperplane. Then condition (3) is equivalent to the condition of extremal aiming<sup>(7, 10)</sup>. Analogous conditions for the stability of absorption are also formulated for the rough nonlinear cases (1), similar to those considered in<sup>(11)</sup>, but they are probably not very effective. The described method of assigning the strategy  $U^0$  by contingencies (3) can also be based on those sets  $\mathcal{W}(t, \vartheta)$  which are given by the constructions described in<sup>(2-6)</sup>, where the control  $u[t]$  is implemented with allowance for  $v[t]$ . The formalization of motions within the framework of differential equations in contingencies then makes it possible to form the control  $u[t]$  without resorting to the indicated discrimination of the second player. Finally, we note that in a substantive sense the formal strategy  $U^0$  is revealed in discrete approximating schemes similar to those described in articles<sup>(10-12)</sup>.

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