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INVARIANT DESCRIPTION OF V. P. MASLOV' S CANONICAL OPERATOR

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In a number of recent works on the quasiclassical approximation in quantum mechanics, V. P. Maslov proposed considering, instead of the usual expansion of the WKB method, a certain more general expression. This expression, which Maslov called the canonical operator, in the case of the Cauchy problem makes it possible to obtain the quasiclassical approximation for all times. In the present note a new formula for Maslov' s canonical operator is given.

1. Let M be the unitary space C^n, regarded as a real space. We denote the scalar product in C^n by

$$\langle\cdot,\cdot\rangle = (\cdot,\cdot) + i[\cdot,\cdot].$$

An n-dimensional subspace in M is called a Lagrangian plane if the form $[\cdot,\cdot]$ vanishes on it. The set of Lagrangian planes will be denoted by Λ (see ([1,2])). Fix $Q \in \Lambda$ and regard Q as a real Euclidean space with scalar product $qp = (q, p)$. Here and everywhere below the letters q and p denote vectors from Q. Let A be a linear transformation of the space Q. Transformations of M having the form $Ax = Aq + JAp$, where $x = q + Jp$ and J is the operator of the complex structure, will be called real. The automorphisms of M are the unitary group $\mathrm{U}(n)$. An arbitrary transformation $U \in \mathrm{U}(n)$ can be represented in the form $U = (\exp J\Phi)r$, where Φ and r are real transformations: symmetric and orthogonal.

The operator $\exp 2J\Phi$ is uniquely determined by U. The set Λ is the homogeneous space $\mathrm{U}(n)/\mathrm{O}(n)$ ([2]). Representing in the form $(\exp J\Phi)r$ the unitary transformation carrying Q into $\Lambda \in \Lambda$, we establish that to each symmetric real Φ there corresponds a Lagrangian plane, by which $\exp 2J\Phi$ is uniquely determined. We shall call Φ the angle between Q and Λ.

The Schrödinger quantization R assigns to $x \in M$ the self-adjoint operator $R(x)$ in the space $L_2(Q)$:

$$R(x) = q\xi + \frac{h}{i}\, p\frac{\partial}{\partial \xi}.$$

Here $x = q + Jp$; ξ is a variable point of Q and $\partial/\partial\xi$ is the gradient. Let G be the inhomogeneous unitary group of transformations of M. The group G acts naturally on the operators R: for $g = \{U, a\} \in G$, $U \in \mathrm{U}(n)$, $a \in M$, we have $g(R) = UR + aE$. (The basic operations in M are transferred to R in accordance with the definition $(R, x) = R(x)$.) This leads to a unitary projective representation V of the group G, whose operators are determined by the relation

$$g(R) = V^*(g)RV(g)$$

up to a complex factor, which, in turn, up to ± 1, can be fixed by the explicit formula

$$V(g) = \exp\frac{i}{h}[a, R]\exp\left(-\frac{i}{2h}[R, \ln UR]\right).$$

2. The quasiclassical operator is defined below by the properties H_1–H_3.

$\mathbf{H}_1$**.** The quasiclassical operator H will initially be defined by its acti-

by its action on expressions of the form

$$\varphi_0(\xi) = u(\xi)\exp\frac{i}{h}S(\xi), \qquad \xi \in Q, \tag{1}$$

where $h > 0$, S is real, and u is complex, by means of the formula

$$\mathbf{H}\varphi_0 = \left[\sum_{k\geq 0}\left(\frac{h}{i}\right)^k \mathcal{L}_k u\right]\exp\frac{i}{h}S. \tag{2}$$

Here $\mathcal{L}_k$ are linear differential operators whose coefficients depend on the derivatives of the function S at the point ξ. The right-hand side in (2) is regarded as a formal power expansion with respect to h. The quantity $\mathbf{H}\varphi$, where φ itself is a formal expansion

$$\varphi = \sum\left(\frac{h}{i}\right)^k u_k \cdot \exp\frac{i}{h}S \tag{3}$$

is defined in the natural way.

$\mathbf{H}_2$**.** It is required that there exist a real function $H(q, p)$ such that

$$\mathcal{L}_0 u = Hu, \qquad \mathcal{L}_1 u = \frac{\partial H}{\partial p}\frac{\partial u}{\partial \xi} + \frac{1}{2}u\frac{d}{d\xi}\frac{\partial}{\partial p}H,$$

where in the right-hand sides of these formulas $q = \xi$ and $p = \partial S(\xi)/\partial\xi$.

By the symbol

$$\text{S. P.} \int_Q h^{-n/2} v \exp \frac{i}{h} f \, d\xi, \tag{4}$$

where f has a unique nondegenerate critical point η, we shall mean the formal expansion

$$\sum \left(\frac{h}{i}\right)^k v_k \exp \frac{i}{h} f(\eta),$$

which arises if the stationary-phase procedure is formally applied to the symbol of the integral in (4).

Let us now consider a formal expression of the form $V(U)\varphi$, where φ is described above, and $V(U) = V(\{U, 0\})$. We introduce the equivalence relation

$$V\varphi = V_1\varphi_1$$

by the formula

$$\varphi = \text{S.P. } V^{-1}V_1\varphi_1.$$

Here it is understood that the operator $V^{-1}V_1$ must be represented as an integral operator.

$\mathbf{H}_3$**.** Define $\mathbf{H}V\varphi$ as $V\mathbf{H}_V\varphi$, where $\mathbf{H}_V$ has property $\mathbf{H}_1$. We require that $\mathbf{H}$ preserve the equivalence relation; then $\mathbf{H}_V$ is determined by the action of $\mathbf{H}$ on φ. In particular, $\mathbf{H}_V$ satisfies the relations $\mathbf{H}_2$, with H_V obtained as the value of the scalar field H in the coordinate system associated with the plane $\Lambda = UQ$.

3. The classical WKB method for the formal equation

$$ih\,\partial\psi/\partial t = \mathbf{H}(t)\psi \tag{5}$$

deals with solutions of the form (3). It is known, however, that in this class the Cauchy problem is solvable only for sufficiently small t. Formal solutions $V\varphi$ may be considered on the same footing as ordinary solutions, but they too possess the indicated shortcoming. Below a more general construction is given for a formal solution of equation (5), valid for arbitrary t. It is described by the expression

$$\psi(\xi, t) = K_{\Gamma,t}^{x,t_0,\gamma} u = \text{S.P.} \int_{E_t} dS\, T_{E,t}^{x_0,t_0,\gamma}(\xi, x) u(x, t). \tag{6}$$

Here Γ is a Lagrangian manifold (see ([1],[2])); E is the universal covering space for Γ; m^t are diffeomorphisms M, depending on the parameter t, corresponding to the Hamiltonian system

$$J\dot{x} = \frac{\partial}{\partial x} H$$

and $E_t = m^{tE}$; x are points of E_t; $x = q + Jp$; $x_0 = q_0 + Jp_0$ is a fixed point of E_{t_0}; $\xi \in Q$; dS is the area element on E_t; γ is a fixed number;

$u = \sum (h/i)^k u_k$; the meaning of the symbol $\widetilde{\text{S.P.}}$ will be clarified later and, finally,

$$T^{x_0,t_0,\gamma}_{E,t} = V(g(x))|0\rangle \exp \frac{i}{h} \left\{ \int_{x,t}^{x,t_0} \frac{1}{2}(p\,dq - q\,dp) - H\,dt - \frac{1}{2} p_0 q_0 + \gamma \right\}$$

$$= \lim_{\varepsilon \downarrow 0} \left(\det \frac{2\pi h}{i} \sin \Phi_\varepsilon \right)^{-1/2} \exp \frac{i}{h} \left\{ -\frac{1}{2} (\operatorname{ctg} \Phi_\varepsilon (\xi - q), \xi - q) + p(\xi - q) + \int_{x_0,t_0}^{x,t} p\,dq - H\,dt + \gamma \right\}. \tag{7}$$

In the last formula $g(x) = \{U(x), x\} \in G$, and $U(x)$ is some transformation $U = (\exp J\Phi) r \in U(n)$ which takes Q into the tangent plane to E at the point x; $V|0\rangle$ is the result of applying V to $\delta(\xi)$, and $\Phi_\varepsilon = \Phi + i\varepsilon$. The ambiguity in the specification of the kernel T is removed if one assigns $V(g(x))$ some value at the point x_0, t_0 and requires continuity of V on $\bigcup_t E_t$.

It is easy to verify the symbolic equality

$$\int_E dS\, T^{x_0,\gamma}_E u = V(U) \int_{E'} dS\, T^{x'_0,\gamma'}_{E'} u', \tag{8}$$

where $E = UE'$, $x = Ux'$, $x_0 = Ux'_0$, $\gamma - \frac{1}{2} p_0 q_0 = \gamma' - \frac{1}{2} p'_0 q'_0$, and $u(x) = u'(x')$.

Introduce an open covering $\{\theta_\alpha\}$ of the manifold E_t and a partition of unity subordinate to it. Correspondingly, the integral (6) is represented by a sum of integrals, and each of the terms is transformed by formula (8) to the form

$$V(U_\alpha) \widetilde{\text{S.P.}} \int dS\, T_\alpha(\xi, x) u_\alpha. \tag{9}$$

One may assume that θ_α is projected one-to-one onto $\Lambda_\alpha = U_\alpha Q$ and that the angle Φ_α between Λ_α and the tangent planes to θ_α is such that $\det \sin \Phi \neq 0$. Then the integral in (9) has the form (4), and the point $\xi = q$ is a nondegenerate critical point of the function f. By $\widetilde{\text{S.P.}} \int dS\, T_\alpha u_\alpha$ in (9) one should understand the result of formal computations according to the stationary-phase method relative to this point. Thus one may associate with Ku the symbolic sum

$$Ku \leftrightarrow \sum_\alpha V(U_\alpha) \varphi_\alpha. \tag{10}$$

The operations of differentiation with respect to t and of applying H to expressions of the form Ku are defined as follows: formally apply these operations to each term of the symbolic sum and transform the resulting symbolic sum again into an integral of the type Ku.

Put $v_k = (dS/dS_0)^{1/2} u_k$, where dS_0 is the element of surface on E obtained from dS by means of the mapping m^t. Substitution of the expression Ku into equation (5) leads to the following result: the total derivative dv_k/dt along the trajectories of the dynamical system m^t is explicitly expressed in terms of $v_0, v_1, \dots, v_{k-1}$, and $dv_0/dt = 0$. We give here the simplest example of the use of an expression of the form Ku in asymptotic problems.

Let the function u be finite on E_t, and let the covering $\{\theta_\alpha\}$ be locally finite. Truncate all the formal series φ_α in (10) at the number N; by this we assign to the symbol Ku a certain numerical value, which we denote by ^{NKu}. We shall say that Ku is an asymptotic expansion of the function ψ as $h \to 0$: $\psi \sim Ku$, if $h^{-N}(\psi - {}^{NKu}) \to 0$ as $h \to 0$ in $L_2(Q)$.

Let $\mathcal{H}$ be a self-adjoint operator in $L_2(Q)$, independent of t. Suppose that the domain of definition of $\mathcal{H}$ contains functions of class C^∞ which, together with their derivatives, decrease at infinity faster than any power. Let φ_0 be such a function, and let $\mathcal{H}\varphi_0$ admit an asymptotic expansion in $L_2(Q)$ of the form (2), where $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfy conditions H_2. Suppose that the operator $\mathcal{H}_V = V\mathcal{H}V^*$ has analogous properties. Then the asymptotic expansion of $\mathcal{H}\varphi_0$ gives rise to the quasiclassical operator H. Examples of such operators are furnished by pseudodifferential operators with decreasing symbol.

Consider the Cauchy problem for the equation

$$ih\,\partial\psi/\partial t = \mathcal{H}\psi \tag{11}$$

with an initial condition of the form $\psi|_{t=0} \sim K_{\Gamma,0}u$. Then, for any finite t, $\psi \sim K_{\Gamma,t}u$.

4. The expression ${}^1Ku$ is in fact equivalent to Maslov' s canonical operator. The description of this operator includes a certain index ind of oriented curves on a Lagrangian manifold. One may give for ind the formula

$$\mathrm{ind} = \lim_{\varepsilon\downarrow 0} \varkappa(\varepsilon), \tag{12}$$

where $\varkappa(\varepsilon)$ is the increment, under continuous variation along the curve, of the function $-\pi^{-1} \arg\det\cos\Phi_\varepsilon$. In the case of closed curves, formula (12) is closely connected with Arnold' s formula for ind [(2)]; the latter follows from formula (12). Maslov' s formula, which describes the dynamics of the canonical operator under a change of t, includes the index of the trajectory $m^t x$, $x \in E_{t_1}$, $t \in [t_1, t_2]$, connected with the Morse index. It is also expressed in terms of Φ by formula (12).

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[1] V. P. Maslov, *Theory of Perturbations and Asymptotic Methods*, Moscow, 1965.
[2] V. I. Arnold, *Functional Analysis and Its Applications*, 1, no. 1 (1967).

Note: Figure translations are in progress. See original paper for figures.