ON ONE BOUNDARY-VALUE PROBLEM FOR A MODEL EQUATION OF MIXED TYPE WITH TWO PERPENDICULAR LINES OF DEGENERATION

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Abstract

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MATHEMATICS

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ON ONE BOUNDARY-VALUE PROBLEM FOR A MODEL EQUATION OF MIXED TYPE WITH TWO PERPENDICULAR LINES OF DEGENERATION

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In the finite simply connected domain Ω, bounded by the arcs $AB : x^2+y^2=1,\ x\geqslant 0,\ y\geqslant 0;\ B^*A^* : x^2+y^2=1,\ x\leqslant 0,\ y\leqslant 0$ and the segments $BB^* : y-x=1,\ y\geqslant 0;\ A^*A : x-y=1,\ y\leqslant 0$, consider the equation of mixed type

$$u_{xx}+\operatorname{sgn}(xy)u_{yy}=0. \tag{1}$$

Let $CO(OD)$ be the segment $-\frac{1}{2}\leqslant x\leqslant 0$ $(0\leqslant x\leqslant \frac{1}{2})$ of the characteristic $x+y=0$ of equation (1); $BA^*(AB^*)$ the segment of the straight line $x=0$ $(y=0)$; $\Omega_1(\Omega_1^*)$ and $\Omega_2(\Omega_2^*)$ the hyperbolic parts of the mixed domain Ω, where $x>0,\ x+y>0$ $(x>0,\ x+y<0)$ and $x<0,\ x+y>0$ $(x<0,\ x+y<0)$, respectively; $\Omega_3(\Omega_3^*)$ the elliptic part of Ω, where $x>0,\ y>0$ $(x<0,\ y<0)$;

$$\Delta=\bigcup_{i=1}^{3}\Omega_i\cup OA\cup OB,\qquad \Delta^*=\bigcup_{i=1}^{3}\Omega_i^*\cup OA^*\cup OB^*.$$

Problem A. It is required to determine a function $u(x,y)$ with the following properties: 1) $u(x,y)\in C(\overline{\Omega})\cap C^{(1,0)}(\Omega\backslash CD)$; 2) $u_x(x,0),\ u_x(0,y),\ u_y(x,0),\ u_y(0,y)$ may tend to ∞ of order less than unity at the points $A,\ B,\ A^*,\ B^*$ and O; 3) $u(x,y)$ is a regular solution in $\Omega\setminus(BA^*\cup AB^*\cup CD)$ of equation (1), satisfying the boundary conditions

$$u\big|_{AB}=\varphi_1(\theta),\quad 0\leqslant\theta\leqslant\pi/2;\qquad u\big|_{B^*A^*}=\varphi_2(\theta),\quad \pi\leqslant\theta\leqslant 3\pi/2; \tag{2}$$

$$u\big|_{BC}=\psi_1(y),\quad \tfrac{1}{2}\leqslant y\leqslant 1;\qquad u\big|_{A^*D}=\psi_2(y),\quad -1\leqslant y\leqslant -\tfrac{1}{2}.$$

With respect to the prescribed functions it is assumed that

$$\varphi_i(\theta) = \sin^2 2\theta\,\overline{\varphi}_i(\theta), \quad i = 1,2; \qquad \overline{\varphi}_1(\theta) \in C(0 \leqslant \theta \leqslant \pi/2),$$

$$\overline{\varphi}_2(\theta) \in C(\pi \leqslant \theta \leqslant 3\pi/2); \tag{3}$$

$$\psi_1(y) \in C(\tfrac{1}{2} \leqslant y \leqslant 1) \cap C^{(2,\alpha)}(\tfrac{1}{2} < y < 1),$$

$$\psi_2(y) \in C(-1 \leqslant y \leqslant -\tfrac{1}{2}) \cap C^{(2,\alpha)}(-1 < y < -\tfrac{1}{2}), \tag{4}$$

where $\psi_1'(y)$ as $y \to \frac{1}{2},\ y \to 1$ and $\psi_2'(y)$ as $y \to -1,\ y \to -\frac{1}{2}$ may tend to ∞ of order lower than 1; $\varphi_1(\pi/2) = \psi_1(1),\ \varphi_2(3\pi/2) = \psi_2(-1)$.

We shall show that the solution of boundary-value problem A exists, is unique, and can be written in explicit form.

$$u(x,y) = u_i(x,y), \quad (x,y) \in \Omega_i; \qquad u(x,y) = u_i^*(x,y), \quad (x,y) \in \Omega_i^*; \quad i = 1,2; \tag{5}$$

$$u_y(x,0) = \begin{cases} \nu_1(x), & 0 < x < 1, \\ \nu_1^*(x), & -1 < x < 0; \end{cases} \qquad u_x(0,y) = \begin{cases} \nu_2(y), & 0 < y < 1, \\ \nu_2^*(y), & -1 < y < 0; \end{cases} \tag{6}$$

$$u(x, x-1) = X_1(x), \qquad \tfrac{1}{2} \leqslant x \leqslant 1;$$

$$u(x, x+1) = X_2(x), \qquad -1 \leqslant x \leqslant -\tfrac{1}{2}, \tag{7}$$

where we shall assume that $X_1(x) \in C^{(2,\alpha)}(1/2 < x < 1)$, $X_2(x) \in C^{(2,\alpha)}(-1 < x < -1/2)$, while $X_1'(x)$ at $x = 1/2,\ x = 1$ and $X_2'(x)$ at $x = -1,\ x = -1/2$ may have singularities of order less than 1 (below it will be seen that $X_1(x)$ and $X_2(x)$ possess these properties).

The solution $u(x,y)$ of problem A in the domain Δ coincides with the solution of the following Gellerstedt boundary-value problem: find a function $u(x,y)$ having the properties: 1) $u(x,y) \in C(\overline{\Delta}) \cap C^{(1,0)}(\Delta)$; 2) $u_x(x,0)$, $u_x(0,y)$, $u_y(x,0)$, $u_y(0,y)$ may have singularities of order less than 1 at the points A, B, and O; 3) $u(x,y)$ is a regular solution of equation (1) in $\Delta \setminus (OA \cup OB)$, satisfying the boundary conditions $u|_{AB} = \varphi_1(\theta)$, $u|_{BC} = \psi_1(y)$, $u|_{DA} = X_1(x)$.

In paper (^{2}) it was proved that there exists a unique solution of the formulated boundary-value problem, which can be written in explicit form, and that the functions (6) and (7) are connected by the relations:

$$\nu_i(x) = \frac{F_i(x)}{2} + \frac{1}{\pi}\int_0^1 N_1(x,t)F_i(t)\,dt - \frac{1}{\pi}\int_0^1 N_2(x,t)F_j(t)\,dt, \qquad (8)$$

$$\nu_i^*(-x) = \frac{F_i^*(-x)}{2} + \frac{1}{\pi}\int_0^1 N_1(x,t)F_i^*(-t)\,dt - \frac{1}{\pi}\int_0^1 N_2(x,t)F_j^*(-t)\,dt, \qquad (9)$$

where

$$N_k(x,t) = t\left(\frac{1-x^4}{1-t^4}\right)^{1/2}\left[\frac{1}{t^2+\beta x^2} - \frac{t^2}{1+\beta t^2x^2}\right], \qquad \beta = \begin{cases} -1, & k=1, \\ 1, & k=2; \end{cases} \qquad (10)$$

$$F_1(x) = 2\frac{d}{dx}\left[X_1\left(\frac{x+1}{2}\right) - g_1(x)\right], \qquad F_2(x) = 2\frac{d}{dx}\left[\psi_1\left(\frac{x+1}{2}\right) - g_2(x)\right]; \qquad (11)$$

$$F_1^*(-x) = 2\frac{d}{dx}\left[g_1^*(-x) - X_2\left(-\frac{x+1}{2}\right)\right],$$

$$F_2^*(-x) = 2\frac{d}{dx}\left[g_2^*(-x) - \psi_2\left(-\frac{x+1}{2}\right)\right]; \qquad (12)$$

$$g_i(x) = \frac{1}{2\pi}\int_0^{\pi/2} \varphi_1(\theta)\left.\frac{\partial G_i}{\partial n}\right|_{|\zeta|=1} d\theta, \qquad g_i^*(-x) = \frac{1}{2\pi}\int_\pi^{3\pi/2} \varphi_2(\theta)\left.\frac{\partial G_i}{\partial n}\right|_{|\zeta|=1} d\theta; \qquad (13)$$

$$G_1(x;\xi,\eta) = \ln|1-\zeta^2x^2| - \ln|\zeta^2-x^2|, \qquad G_2(x;\xi,\eta) = \ln|1+\zeta^2x^2| - \ln|\zeta^2+x^2|,$$

$$\zeta = \xi + i\eta; \quad 0 < x < 1, \quad i,j = 1,2, \quad i \neq j; \quad n \text{ is the inner normal.}$$

Relations (9) are obtained in the same way as (8)—by solving the Gellerstedt problem in the domain Δ^* with the data $\varphi_2(\theta)$, $\psi_2(y)$, and $X_2(x)$ on B^*A^*, A^*D, and B^*C, respectively.

By virtue of the continuity of the solution $u(x,y)$ of problem A in the closed domain $\overline{\Omega}$, we have

$$u_1(x,-x)=u_1^*(x,-x), \qquad u_2(x,-x)=u_2^*(x,-x), \qquad u_1(0,0)=u_2^*(0,0); \tag{14}$$

$$X_1(1/2)=\psi_2(-1/2), \qquad X_2(-1/2)=\psi_1(1/2). \tag{15}$$

From d' Alembert' s formula it is obvious that the functions u_1 and u_2 (u_1^* and u_2^*) can be written explicitly in terms of the functions X_1,ν_1,ψ_1,ν_2 (respectively ψ_2,ν_2^*, X_2,ν_1^*). Hence, on the basis of equalities (14) and (15), we conclude that

$$\nu_1(x)+\nu_2^*(-x)+\frac{d}{dx}\left[\psi_2\Big(-\frac{x+1}{2}\Big)-X_1\Big(\frac{x+1}{2}\Big)\right]=0,$$

$$\nu_2(x)+\nu_1^*(-x)+\frac{d}{dx}\left[X_2\Big(-\frac{x+1}{2}\Big)-\psi_1\Big(\frac{x+1}{2}\Big)\right]=0, \tag{16}$$

$$2\left[\psi_2(-1/2)-\psi_1(1/2)\right]+X_2(-1)-X_1(1)+\int_0^1\left[\nu_1(t)+\nu_1^*(-t)\right]dt=0.$$

Eliminating from (8), (9), and (16) $v_i(x)$, $v_i^*(-x)$, $i=1,2$, and taking into account (11), (12), we obtain

$$\frac{2}{\pi}\int_0^1 N_1(x,t)\frac{d}{dt}\left[X_1\left(\frac{t+1}{2}\right)\right]dt+\frac{2}{\pi}\int_0^1 N_2(x,t)\frac{d}{dt}X_2\left(-\frac{t+1}{2}\right)dt=P_1(x), \tag{17}$$

$$\frac{2}{\pi}\int_0^1 N_1(x,t)\frac{d}{dt}X_2\left(-\frac{t+1}{2}\right)dt+\frac{2}{\pi}\int_0^1 N_2(x,t)\frac{d}{dt}X_1\left(\frac{t+1}{2}\right)dt=P_2(x), \tag{18}$$

$$\frac{2}{\pi}\int_0^1 ds\int_0^1 N_1(s,t)\frac{d}{dt}\left[X_1\left(\frac{t+1}{2}\right)-X_2\left(-\frac{t+1}{2}\right)\right]dt=P, \tag{19}$$

where

$$P_1(x) = \frac{d}{dx}\left[g_1(x) - g_2^*(-x)\right] + \frac{1}{\pi}\int_0^1 N_1(x,t)h_1(t)dt + \frac{1}{\pi}\int_0^1 N_2(x,t)h_2(t)dt, \tag{20}$$

$$P_2(x) = \frac{d}{dx}\left[g_1^*(-x) - g_2(x)\right] + \frac{1}{\pi}\int_0^1 N_1(x,t)h_2(t)dt + \frac{1}{\pi}\int_0^1 N_2(x,t)h_1(t)dt, \tag{21}$$

$$h_1(x) = 2\frac{d}{dx}g_1(x) - F_2^*(-x), \qquad h_2(x) = 2\frac{d}{dx}g_1^*(-x) + F_2(x), \tag{22}$$

and P is a particular number which depends only on the prescribed functions ψ_1, ψ_2, φ_1, and φ_2.

Consequently, problem A has been reduced to finding functions $X_1\left(\frac{x+1}{2}\right)$, $X_2\left(-\frac{x+1}{2}\right) \in C^{(2,\alpha)}(0 < x < 1)$, satisfying the system of equalities (17), (18), (19) and the conditions (15), where the first derivatives of X_1 and X_2 may become infinite of order less than unity at $x = 0$ and $x = 1$.

It is clear that the functions X_1 and X_2 satisfying equalities (17) and (18) can be obtained from the solution of the equations

$$\frac{2}{\pi}\int_0^1 [N_1(x,t) \pm N_2(x,t)]\mu_i(t)dt = M_i(x), \qquad i = 1, 2, \tag{23}$$

where

$$\mu_i(x) = \frac{d}{dx}\left[X_1\left(\frac{x+1}{2}\right) \pm X_2\left(-\frac{x+1}{2}\right)\right], \qquad M_i(x) = P_1(x) \pm P_2(x), \tag{24}$$

with the plus sign taken for $i = 1$, and the minus sign for $i = 2$.

Substituting the values of $N_1(x,t)$, $N_2(x,t)$ from (10) into (23), after the change of variables

$$t = \tau^{1/4}\left(1 + \sqrt{1-\tau^2}\right)^{-1/4}, \qquad x = y^{1/4}\left(1 + \sqrt{1-y^2}\right)^{-1/4}, \tag{25}$$

equations (23) take the form [2]

$$\frac{1}{\pi}\int_0^1 \frac{\rho_i(\tau)}{\tau - y}d\tau = R_i(y), \qquad i = 1, 2, \tag{26}$$

where

$$\rho_i(y) = \delta_i(x)\mu_i(x), \qquad R_i(y) = \delta_i(x)M_i(x); \tag{27}$$

$$\delta_1(x) = (1+x^8)(1+x^4)^{-1}(1-x^4)^{-1/2}, \qquad \delta_2(x) = (1+x^8)x^{-2}(1-x^4)^{-1/2}, \tag{28}$$

and x is related to y by equality (25).

It follows from (13) that $g_i(x)$ and $g_i^*(-x)$, for $0 < x < 1$, are analytic functions.

Taking (3) into account, we ascertain that $\frac{d}{dx}g_i(x)$, $\frac{d}{dx}g_i^*(-x)$ have finite limits as $x \to 0$, $x \to 1$. Hence, by virtue of the assumptions on the given functions $\psi_i(x)$ (see (4)), the equalities (11), (12), (20), (21), (22), (24), and the properties of integrals of Cauchy type [1], we obtain that $M_i(x) \in C^{(1,\alpha)}(0 < x < 1)$, $i = 1, 2$; moreover, at $x = 0$ and $x = 1$ these functions may tend to ∞ with order less than 1.

On the basis of equalities (25), (27), and (28) we now conclude that the functions $R_i(y) \in C^{(1,\alpha)}(0 < y < 1)$ and may tend to ∞ with order less than 1/4 (3/4) in the case $i = 1$, and less than 3/4 (3/4) in the case $i = 2$, when $y \to 0$ ($y \to 1$). From the same relations (24), (27), and (28) it follows that the solutions of equations (26), $\rho_i(y)$, on the interval $0 \leq y \leq 1$ must possess the same properties as $R_i(y)$. It is known that such solutions of equations (26) exist and can be written in quadratures [1].

Solving (26) and returning to the functions X_1, X_2, by virtue of equality (19) and conditions (15), we ascertain that X_1 and X_2 are determined uniquely. Consequently, problem A is uniquely solvable.

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CITED LITERATURE

1. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.
2. M. M. Zainulabidov, *Differential Equations*, 5, No. 1 (1969).

Note: Figure translations are in progress. See original paper for figures.

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