ON THE SOLUTION OF ITERATED CAUCHY PROBLEMS IN BASIS SERIES

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Abstract

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MATHEMATICS

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ON THE SOLUTION OF ITERATED CAUCHY PROBLEMS IN BASIS SERIES

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Denote by $u(x, s; a, b, c)$ the solution of the Cauchy problem

$$L[u] \equiv \left(D^2 + \frac{a}{s}D + b^2 - X\right)^m u - c^{2m}u = 0, \qquad m = 1, 2, \dots; \tag{1a}$$

$$D^{2m-2}u\big|_{s=0} = \tau(x), \qquad D^k u\big|_{s=0} = 0, \quad k = 0, 1, \dots, 2m-3;\ 2m-1, \tag{1b}$$

where $s \geq 0$; $D = \partial/\partial s$; $a > 0$, $b, c = \text{const}$; X is a linear operator independent of s, acting on the variables $x = (x_1, \dots, x_n)$. Comparing $u(x, s)$ with its value $z(x, s; a, b)$ for $m = 1$, $c = 0$, we find:

1. Let $a = 2\beta$, $\beta_0 = \beta_2 - \beta_1$, $2\nu_k = a_k - 1$, $\gamma = \beta_0 + m(N+1) - 1 > 0$, $(m)_{mk}(\nu + m)_{mk}2^{2mk}A_k(\nu) = (cs)^{2mk}$. Then

$$u(x, s; a_2, b, c) = \frac{s^{2m-2}}{(2m-2)!}\sum_{k=0}^{N-1} A_k(\nu_2)\, z\left[x, s; a_2 + 2(mk+m-1), b\right] + R_N; \tag{2a}$$

$$R_N = \delta_1 s^{2m-2}\int_0^1 \xi^{a_1}(1-\xi^2)^{\gamma-1}Q(\xi)\, z(x, \xi s; a_1, b)\, d\xi; \tag{2b}$$

$$Q = A_N(\beta_0 - 1)\, {}_1F_{2m}\left[1;\, p_i, q_i;\, (cs\sqrt{1-\xi^2}/2m)^{2m}\right], \qquad i = 0, 1, \dots, m-1;$$

$$p_i = N+1+i/m, \qquad mq_i = \gamma + i, \qquad (2m-2)!\Gamma(\nu_1+1)\Gamma(\beta_0+m-1)\delta_1 = 2\Gamma(\nu_2+m).$$

If $|z(x, s; a_1, b)| \leq M$, then $\lim\limits_{N\to\infty} R_N = 0$, and (2a), as $N \to \infty$, gives

$$u(x,s;a,b,c)=\sum_{k=0}^{\infty}A_k(\nu)\,u(x,s;a+2mk,b,0); \tag{3a}$$

$$(2m-2)!\,u(x,s;a,b,0)=s^{2m-2}z[x,s;a+2(m-1),b]. \tag{3b}$$

2. For $a_2+2(m-1)>a_1\geq 0$, $b_0=\sqrt{b_2^2-b_1^2}$, $\gamma_1=\beta_0+m(k+1)-1$:

$$u(x,s;a_2,b_2,c)=\delta_1 s^{2m-2}\int_0^1\xi^{a_1}(1-\xi^2)^{\beta_0+m-2}T(\xi)\,z(x,\xi s;a_1,b_1)\,d\xi, \tag{4a}$$

$$T(\xi)=\sum_{k=0}^{\infty}A_k(\beta_0-1)(1-\xi^2)^{km}\,\overline{I}_{\gamma_1-1}\left(b_0 s\sqrt{1-\xi^2}\right). \tag{4b}$$

When $b_2=b_1=b$, (4b) reduces to the normalized Bessel function of order $2m$:

$$T=\Omega\left[\beta_0-1;\ (cs\sqrt{1-\xi^2}/2m)^{2m}\right],$$

where

$$\Omega[a;\Lambda]={}_0F_{2m-1}[\mu_i,\nu_j;\Lambda],\qquad \mu_i=1+i/m,\quad \nu_j=1+(a+j)/m, \tag{5}$$

with $i=1,\dots,m-1$; $j=0,1,\dots,m-1$. For $b_2\neq b_1$, for the generalized Humbert function of higher order $T(\xi)$, other expressions are found–

···(series in powers of b_0, integral representations). If $\tau(x)\in C^\infty$, then

$$u(x,s;a,b,c)=\frac{s^{2m-2}}{(2m-2)!}\sum_{k=0}^{\infty}B_k(s)(X-b^2)^k\tau(x), \tag{6}$$

$$k!(\nu+m)_k 2^{2k}B_k=s^{2k}\Omega[\nu+k;(cs/2m)^{2m}].$$

3. Formulas (2), (3), (4), (8), (9), (15) from (1) and (3), (4) give basis series of other types for $u(x,s)$. For example, for $m=2$, $\lambda=$ const,

$$c^2A_k=(\nu_2+1)g_k(\nu_1)[\Lambda_k^{(2)}-\Lambda_k^{(1)}]:$$

$$u(x,\lambda s;a_2,b,c)=\sum_{k=0}^{\infty}A_{ks}^{2k}(X-b^2)^{kz}(x,s;a_1+2k,b), \tag{7}$$

$$\Lambda_k^{(l)}=\overline{\mathrm{E}}_2[-k,\nu_1+1,\nu_2+1;\lambda^2,(-1)^l(c\lambda s/2)^2],\qquad l=1,2.$$

Let us also note the expansion ($m = 2$; see g_k and $\bar{g}_k$ in (1))

$$u(x, s; a, b, c) = \sum_{k=0}^{\infty} \bar{A}^{2k}_{ks}(X - b^2)^{kz}(x, s; a + 4k, b), \tag{8}$$

$$\bar{A}_k = \alpha^{2k-2}_{kc} s^{2k}[\bar{I}_{\nu+2k}(cs) - (-1)^k \bar{J}_{\nu+2k}(cs)],$$

where $(\nu + 1)_{2k} 2^{4k} \alpha_k = (\nu + 1)\bar{g}_k(\nu)$. Replacing in (1) and (3) a, s, c by $2\varepsilon - 1, 2\sqrt{\varepsilon s}, \sqrt{c}$, we obtain, as $\varepsilon \to \infty$, $L_1 \equiv (D + b^2 - X)^m - c^m$

$$L_1[v] = 0, \quad D^{m-1}v\big|_{s=0} = \tau(x), \quad D^{kv}\big|_{s=0} = 0, \quad k = 0, 1, \dots, m-2; \tag{9}$$

$$v(x, s; b, c) = \lim_{\varepsilon \to \infty} u(x, 2\sqrt{\varepsilon s}; 2\varepsilon - 1, b, \sqrt{c}) = H(s)v(x, s; b, 0); \tag{10a}$$

$$(m - 1)!v(x, s; b, 0) = s^{m-1}w(x, s; b). \tag{10b}$$

Here w is the value of v for $m = 1,\ c = 0$, and $H(s)$ is the normalized hyperbolic function of order m:

$$H(s) = {}_0F_{m-1}[\mu_i; (cs/m)^m] = (m - 1)!(cs)^{1-m} h_m(cs, m). \tag{10c}$$

Replace in (7) $\lambda s, a_1$ by $2\sqrt{\lambda s}, 2\varepsilon - 1$, and let $\varepsilon \to \infty$; then

$$u(x, 2\sqrt{\lambda s}; a, b, c) = \left(\frac{\nu + 1}{c^2}\right) \sum_{k=0}^{\infty} \tilde{A}^{k}_{ks}(X - b^2)^{kv}(x, s; b), \tag{11}$$

$$k!\tilde{A}_k = (-1)^k[\Lambda^{(2)}_k - \Lambda^{(1)}_k], \qquad \Lambda^{(l)}_k = \Phi_3[-k, \nu + 1; \lambda, (-1)^{lc^{2\lambda}} s].$$

The convergence of the series (7), (8), (11) is proved with the aid of estimates obtained from the integral representations of the quantities $A_k, \bar{A}_k$, and $\tilde{A}_k$.

4. We give examples of the application of basis series to other problems for (1a) and (9a). Substituting (3a), (6), (7), (8), (11) into

$$\bar{u}(x, s; a, b, c) = (2m - 1)^{-1} s^{1-a} u(x, s; 2 - a, b, c), \tag{12}$$

we obtain the solution $\bar{u}(x, s; a, b, c)$ of the Cauchy problem $L[\bar{u}] = 0$,

$$D^{2m-1}(s^a \bar{u})\big|_{s=0} = \tau(x), \qquad D^k \bar{u}\big|_{s=0} = 0, \qquad k = 0, 1, \dots, 2m-2. \tag{13}$$

Another integral $u_1(x,s;a,b,c)$ of equation (1a) with data

$$u_1(x,0)=\tau(x), \qquad D_1^{ku}\big|_{s=0}=0, \qquad k=1,\dots,2m-1, \tag{14}$$

arises if one introduces into (3a) the finite basis sum

$$u_1(x,s;a,b,0)=\sum_{k=0}^{m-1} g_k(\nu)s^{2k}(X-b^2)^{kz}(x,s;a+2k,b). \tag{15}$$

For (14) one constructs equalities, analogous to (2), (4), (5), relating $u_1(x,s;a_2,b_2,c)$ to $u_1(x,s;a_1,b_1,0)$.

We also note the confluent case of problem (1a), (14):

$$L_1[v_1]=0, \qquad v_1\big|_{s=0}=\tau(x), \qquad D^k v_1\big|_{s=0}=0, \quad k=1,\dots,m-1, \tag{16}$$

for which (15), after the limiting transition (10a), gives

$$v_1(x,s;b,c)=H(s)\sum_{k=0}^{m-1}\frac{(-1)^k}{k!}\,s^k\,(X-b^2)^k w(x,s;b). \tag{17}$$

The expressions (15) and (17) (for $c=0$) are partial sums of the inversion formulas (see (1.10a) from (2)), and therefore, as $m\to\infty$, these sums converge to $\tau(x)$. The equalities

$$(X-b^2)^k z(x,s;a+2k,b)=2^{2k}(\nu+1)_k D_{s^2}^k z(x,s;a,b), \tag{18}$$

and $(X-b^2)^k w = D^k w$ transform (7), (15) and (11), (17) into expansions in powers of the operators $D_{s^2}=\partial/\partial(s^2)$ and $D=\partial/\partial s$. If $n=1$, $X=\partial/\partial x$, and the basis $z(x,s)$ of the series (3) has the data $z(x,0)=\tau(x)$, $z(0,s)=0$, $\tau(0)=0$, $x\geq 0$, $s\geq 0$, then (3) gives a solution $\tilde{u}(x,s)$ of the mixed boundary-value problem:

$$D^{2m-2}\tilde{u}\big|_{s=0}=\tau(x), \qquad \tilde{u}\big|_{x=0}=0, \qquad D^k\tilde{u}\big|_{s=0}=0, \quad k=0,1,\dots,2m-3. \tag{19}$$

For $\tilde{u}$ there also hold relations of the form (2), (4). For example,

$$u(x,s;a_2,b,c)=\delta_2 s^{2m-2}\int_1^\infty \xi^{a_1}(\xi^2-1)^{\beta_1+m-2}T(\xi)\,z(x,\xi s;a_1,b)\,d\xi, \tag{20}$$

$$\beta_0 = \beta_2 - \beta_1 > 1 - m,$$

$$(2m-2)!\Gamma(\beta_0 + m - 1)\Gamma(1 - \nu_2 - m)\delta_2 = 2\Gamma(-\nu_1),$$

where $T(\xi)$ is the function (5) for $\alpha = \beta_0 - 1$, $\Lambda = (-1)^m (cs\sqrt{\xi^2 - 1}/2m)^{2m}$. Replacing the bases of the expansions (3a), (7), (8), (11), (15), (17) by their integral representations from (1), we arrive at new relations for u, v, w, z. For example, (19) from (1) and (17) give

$$v_1(x, s; b, c) = \frac{H(s)}{\Gamma(\nu+1)} \int_0^\infty \lambda^\nu e^{-\lambda} L_{m-1}^{(\nu+1)}(\lambda)\, z(x, 2\sqrt{\lambda}\, s; a, b)\, d\lambda. \qquad (21)$$

Inverting (4a), (20) and (21) with respect to z, one can construct transformation operators taking $u(a_1, b_1, c_1)$, $v(b_1, c_1)$ into $u(a_2, b_2, c_2)$, $v(b_2, c_2)$.

5. Let

$$X = \sum_{k=1}^{n} \frac{\partial}{\partial x_k},$$

then

$$w = e^{-b^2 s} \tau(x_1 + s, \dots, x_n + s),$$

and here (10), (11) and (17) determine the solutions v, u, v_1. If

$$X = \Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2},$$

then $z(x, s; n-1, 0) = M[x, s; \tau(x)]$, and therefore (4a) for $a_1 = n - 1$, $b_1 = 0$; (7), (18) for $a_1 = n - 1$, $b = 0$; (15), (18) for $a = n - 1$, $b = 0$ turn into explicit resolving operators (cf. (3-5)). For example, when $n = 1$, $X = \partial^2/\partial x^2$, $a_1 = b_1 = b_2 = 0$, $\beta + m > 1$, from (4a) it follows that

$$u(x, s; a, 0, c) = \bar{\delta}_1 s^{2m-2} \int_0^1 \tau[x + s(2t-1)]\, [t(1-t)]^{\beta+m-2} T_1(t)\, dt,$$

$$\sqrt{\pi}(2m-2)!\Gamma(\beta + m - 1)\bar{\delta}_1 = 2^{a+2m-3}\Gamma(\nu + m), \qquad T_1 = \Omega(\beta - 1; \Lambda),$$

where $\Lambda = [cs\sqrt{t(1-t)}/m]^{2m}$. For (1a), (19), proceeding from (3), we obtain

$$\tilde{u}(x,s;a,b,c) = \varkappa_1 s^{1-a} \int_0^x \tau(\xi)(x-\xi)^{\beta+m-5} H_1(\xi) \exp\left[b^2(x-\xi) - \frac{s^2}{4(x-\xi)}\right] d\xi,$$

$$(2m-2)!\Gamma(1-\nu-m)\chi_1 = 2^{a+2m-3}, \qquad H_1 = {}_0F_{m-1}\left[\mu_i; \left[-c^2(x-\xi)/m\right]^m\right],$$

where the μ_i are the same as in (5) and (10c). If $n = 1$, $X = \partial^2/\partial x^2$, and for the basis $z(x,s)$ of the series (3a) $z(x,0) = \tau(x)$, $z(x,x) = 0$, $\tau(0) = 0$, then (3a) solves the singular Tricomi problem

$$D^{2m-2}u\big|_{s=0} = \tau(x), \qquad u\big|_{s=x} = 0, \qquad D^k u\big|_{s=0} = 0, \qquad k = 0, 1, \dots, 2m-3. \tag{22}$$

Here, for $b = 0$, $\beta + m > 1$,

$$(2m-2)!\Gamma(\beta+m-1)\Gamma(1-\nu-m)\chi_2 = 2\sqrt{\pi},$$

$$u(x,s;a,0,c) = \chi_2 s^{1-a} \int_0^{x-s} r^{a+2m-4} T_2(\xi)\tau(\xi)\, d\xi,$$

$$T_2 = \Omega\left[\beta-1;\, (-1)^m (cr/2m)^{2m}\right], \qquad r = \sqrt{(x-\xi)^2 - s^2}.$$

Using, as the basis of expansions (3a), (12), (15), the expressions (7a) and (13a) from (6), one obtains solutions of problems (1), (13), (14), (22) when $n = 1$, and X is the Bessel operator

$$X = \frac{\partial^2}{\partial x^2} + \frac{2\mu}{x}\frac{\partial}{\partial x}.$$

In this case one constructs Riemann, Green-Hadamard functions and fundamental (elementary) solutions of equation (1a) in the form of integrals and series, convergent with (3)–(6) from (6), and with their aid more general iterated initial and boundary-value problems are investigated. For example, for $c = 0$,

$$X = \Delta + \frac{2\mu}{x_n}\frac{\partial}{\partial x_n},$$

by analogy with (6a), (6c) from (6), we find $2N = 2m-n+1$, $2\gamma = a+n-2m+1$,

$$U = (ss_0)^{-\beta}(x_{nx}n^0)^{-\mu}R^{2N-2}\sum_{k=0}^{\infty}\frac{\rho^k}{(N)_k\,k!}F_3(\beta,\mu,1-\beta,1-\mu,N+k;\omega,\lambda),$$

$$\overline{V} = (x_{nx}n^0)^{-\mu}R^{-2\gamma}\sum_{k=0}^{\infty}\frac{\rho^k}{(1-\gamma)_k k!}H_2\left(\gamma-k,\beta,\mu,1-\mu,\alpha;\frac{1}{\omega},-\lambda\right),$$

$$R^2=\sum_{i=1}^{n}(x_i-x_i^0)^2-(s-s_0)^2, \qquad 4\rho=b^2R^2, \qquad 4ss_0\omega=R^2, \qquad 4x_{nx}n^0\lambda=-R^2,$$

where F_3 and H_2 are hypergeometric functions of Appell and Horn.

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Note: Figure translations are in progress. See original paper for figures.

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