ON RIEMANNIAN SPACES ADMITTING A GENERALLY RECURRENT SYMMETRIC TENSOR OF THE SECOND ORDER

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Abstract

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ON RIEMANNIAN SPACES ADMITTING A GENERALLY RECURRENT SYMMETRIC TENSOR OF THE SECOND ORDER

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1°. Definition. A symmetric tensor b_{ij} of the second order will be called **generally recurrent** if it satisfies the condition

$$b_{ij,l} = \bar{\lambda}_l g_{ij} + \lambda_i g_{jl} + \lambda_j g_{il} + \bar{\mu}_l b_{ij} + \mu_i b_{jl} + \mu_j b_{il}, \tag{1}$$

where g_{ij} is the metric tensor of the Riemannian space V_n; $\bar{\lambda}_i, \lambda_i, \bar{\mu}_i, \mu_i$ are certain covariant tensors, which we shall call the **tensors of generalized recurrence**; the comma denotes covariant differentiation.

In particular, if $\bar{\lambda}_i = 0, \lambda_i = 0, \mu_i = 0$, then we obtain a recurrent tensor b_{ij} ([1]). Condition (1) also includes the tensor characteristic of hypersurfaces of the second order ([2]) (for $\bar{\lambda}_i = 0, \lambda_i = 0, \bar{\mu}_i = \mu_i$), and the condition that the metrics g_{ij} and b_{ij} have common geodesics ([3]) (for $\lambda_i = 0, \bar{\lambda}_i = 0, \bar{\mu}_i = 2\mu_i$), and, finally, the characteristic condition that the vector field ξ_i determines a one-parameter group of projective transformations ([3]) (for $\bar{\mu}_i = 0, \mu_i = 0, \bar{\lambda}_i = 2\lambda_i,\ b_{ij} = \xi_{(i,j)}$).

In the present article we find all Riemannian spaces admitting a generally recurrent symmetric tensor of the second order, i.e., we find all metrics for which the system of equations (1) is compatible.

2°. Preliminary information. Among the eigenvalues $k_1, k_2, \dots, k_n$ of the tensor b_{ij}, i.e., among the roots of the equation $|kg_{ij} - b_{ij}| = 0$, there may also be multiple ones. Let a frame of eigenvectors $\eta^i_{a_1}$ be orthonormalized: $g_{ij}\eta^i_{a_1}\eta^j_{b_1} = \delta_{ab}$; then we also have $b_{ij}\eta^i_{a_1}\eta^j_{b_1} = k_a\delta_{ab}$, so that from (1) we obtain

$$b_{ij,l}\eta^i_{a_1}\eta^j_{b_1}\eta^l_{c_1} = 0, \qquad a, b, c \neq . \tag{2}$$

When condition (2) is fulfilled, as is known ([3]), one can pass to new variables in such a way that we shall have

$$g_{ij}du^i du^j = \sum_{\alpha=1}^{p} \Phi_\alpha, \qquad b_{ij}du^i du^j = \sum_{\alpha=1}^{p} k_\alpha \Phi_\alpha, \tag{3}$$

where $k_1, \dots, k_p$ are distinct roots, and the form Φ_α contains only the differentials of the variables u^{i_α} corresponding to k_α. The variables $u^1, u^2, \dots, u^n$ are divided into p groups $u^{i_1}, \dots, u^{i_p}$ (according to the number of distinct roots), and the number of variables in each group is equal to the multiplicity of the corresponding root, for example $u^{i_1} \equiv (u^1, \dots, u^{m_1})$, where m_1 is the multiplicity of the root k_1. If among the roots $k_1, \dots, k_n$ there are simple ones, then we write them in the first places. Thus, if k_α is a simple root, then the group of variables u^{i_α} consists of the single variable u^α.

Equalities (3) mean that

$$g_{i_\alpha j_\beta} = b_{i_\alpha j_\beta} = 0,\ \alpha \neq \beta; \qquad b_{i_\alpha j_\alpha} = k_\alpha g_{i_\alpha j_\alpha}, \tag{4}$$

where $g_{i_\alpha j_\alpha}$, $b_{i_\alpha j_\alpha}$ depend, generally speaking, on all variables $u^1, \dots, u^n$.

3°. **Basic equations.** Computing directly the derivative $b_{i_\alpha j_\alpha, l_\beta}$, and also from (1) (using in both cases (4)), we find

$$(k_\alpha - k_\beta)\partial \ln |g_{j_\beta, l_\beta}| / \partial u^{i_\alpha} = 2\lambda_{i_\alpha} + 2k_\beta \mu_{i_\alpha}. \tag{5}$$

Similarly, computing $b_{j_\beta l_\beta, i_\alpha}$ and $b_{i_\alpha i_\alpha, i_\alpha}$, we obtain

$$\partial k_\beta / \partial u^{i_\alpha} = \bar{\lambda}_{i_\alpha} + k_\beta \bar{\mu}_{i_\alpha}, \tag{6}$$

$$\partial k_\alpha / \partial u^{i_\alpha} = \bar{\lambda}_{i_\alpha} + k_\alpha \bar{\mu}_{i_\alpha} + 2\lambda_{i_\alpha} + 2k_\alpha \mu_{i_\alpha}.$$

Computing $b_{i_\alpha i_\alpha, j_\alpha}$, using (1), (4), and (7), we have

$$g_{i_\alpha i_\alpha}(\lambda_{j_\alpha} + k_\alpha \mu_{j_\alpha}) - g_{i_\alpha j_\alpha}(\lambda_{i_\alpha} + k_\alpha \mu_{i_\alpha}) = 0. \tag{7}$$

Interchanging the indices i_α and j_α and using the positive definiteness of the form $g_{ij}du^i du^j$, we find

$$\lambda_{i_\alpha} + k_\alpha \mu_{i_\alpha} = 0, \quad \text{if } k_\alpha \text{ is multiple.} \tag{8}$$

Consequently,

$$\partial k_\alpha / \partial u^{i_\alpha} = \bar{\lambda}_{i_\alpha} + k_\alpha \bar{\mu}_{i_\alpha}, \quad \text{if } k_\alpha \text{ is multiple.} \tag{9}$$

From (6), replacing β by γ and subtracting from (6), we obtain

$$\partial \ln |k_\beta - k_\gamma| / \partial u^{i_\alpha} = \bar{\mu}_{i_\alpha}. \tag{10}$$

From (10), replacing γ by δ and subtracting from (10), we obtain

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \bigl|(k_\beta - k_\gamma)/(k_\beta - k_\delta)\bigr| = 0, \quad \alpha, \beta, \gamma, \delta \neq . \tag{11}$$

Here and below $p \geq 4$. Taking the logarithm and differentiating with respect to u^{l_β} of the identity

$$\frac{k_\alpha - k_\beta}{k_\alpha - k_\gamma} = \frac{k_\alpha - k_\beta}{k_\alpha - k_\delta} \frac{k_\alpha - k_\delta}{k_\alpha - k_\gamma}$$

and taking (11) into account, we find

$$\frac{\partial}{\partial u^{l_\beta}} \ln \left| \frac{k_\alpha - k_\beta}{k_\alpha - k_\gamma} \right| = \frac{\partial}{\partial u^{l_\beta}} \ln \left| \frac{k_\alpha - k_\beta}{k_\alpha - k_\delta} \right| \equiv \varphi_{\alpha l_\beta}(u^{i_\alpha}, u^{l_\beta}). \tag{12}$$

4°. **The functions $\varphi_{\beta i_\alpha}$.** Eliminating $\bar{\lambda}_{i_\alpha}$ from (6) and (9), we obtain

$$\partial \ln |k_\beta - k_\alpha| / \partial u^{i_\alpha} = \bar{\mu}_{i_\alpha}, \quad k_\alpha \text{ multiple.} \tag{13}$$

From (10) and (13) we find

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{k_\beta - k_\alpha}{k_\beta - k_\gamma} \right| = 0, \quad \text{if } k_\alpha \text{ is multiple,}$$

i.e., we have $\varphi_{\beta i_\alpha} = 0$, if k_α is multiple. Let $\varphi_{\beta\alpha} \neq 0$. Computing the ratio $\varphi_{\gamma\alpha} : \varphi_{\beta\alpha}$ by means of (12) and replacing the derivatives of $k_\alpha, k_\beta, k_\gamma$ by formulas (6) and (7), we obtain

$$\varphi_{\gamma\alpha}/\varphi_{\beta\alpha} = (k_\alpha - k_\beta)/(k_\alpha - k_\gamma). \tag{14}$$

Substituting (14) into (12), we find

$$\frac{\partial \varphi_{\beta\alpha}}{\partial u^{l_\beta}} = -\varphi_{\beta\alpha}\varphi_{\alpha l_\beta}. \tag{15}$$

The solution of the system (15) is the functions

$$\varphi_{\beta\alpha} = \varphi'_\alpha/(\varphi_\alpha - \varphi_\beta), \tag{16}$$

where φ_α is an arbitrary function of u_α, and $\varphi_\alpha = \text{const}$ if k_α is multiple. Thus the functions $\varphi_{\beta i_\alpha}$ are found from (16) both in the case when k_α, k_β are multiple, and when they are simple.

5°. **The curvatures k_α.** From (14) and (16) we obtain

$$(k_\alpha - k_\beta)/(\varphi_\alpha - \varphi_\beta) = (k_\alpha - k_\gamma)/(\varphi_\alpha - \varphi_\gamma), \tag{17}$$

i.e. the ratio $(k_\alpha - k_\beta)/(\varphi_\alpha - \varphi_\beta)$ does not depend on β. But

$$(k_\alpha - k_\beta)/(\varphi_\alpha - \varphi_\beta) = (k_\beta - k_\alpha)/(\varphi_\beta - \varphi_\alpha),$$

i.e., this ratio does not depend on α, and we shall denote it by F:

$$(k_\alpha - k_\beta)/(\varphi_\alpha - \varphi_\beta) = F. \tag{18}$$

From (18) we find $k_\alpha - \varphi_\alpha F = k_\beta - \varphi_\beta F \equiv \Phi$, or

$$k_\alpha = \varphi_\alpha F + \Phi, \tag{19}$$

where F and Φ are arbitrary functions.

6°. **Metric.** Adding (5), (6) and subtracting (7), we obtain

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{g_{j_\beta l_\beta}}{g_{j_\gamma l_\gamma}} \right| = -\bar{\mu}_{i_\alpha} - 2\mu_{i_\alpha}. \tag{20}$$

From (20), replacing β by γ and subtracting from (20), we obtain

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{g_{j_\beta l_\beta}}{g_{j_\gamma l_\gamma}} \right| = \frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{k_\alpha - k_\beta}{k_\alpha - k_\gamma} \right|. \tag{21}$$

From (17) and (21) we find

$$\frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{g_{j_\beta l_\beta}}{g_{j_\gamma l_\gamma}} \right| = \frac{\partial}{\partial u^{i_\alpha}} \ln \left| \frac{\varphi_\alpha - \varphi_\beta}{\varphi_\alpha - \varphi_\gamma} \right|, \qquad \alpha \neq \beta, \gamma. \tag{22}$$

Integrating (22), we have

$$\frac{g_{j_\beta l_\beta}}{g_{j_\gamma l_\gamma}} = B_{j_\beta l_\beta j_\gamma l_\gamma} \prod_{\sigma \neq \beta, \gamma} \left| \frac{\varphi_\sigma - \varphi_\beta}{\varphi_\sigma - \varphi_\gamma} \right|, \tag{23}$$

where $B_{j_\beta l_\beta j_\gamma l_\gamma}$ are functions only of $u^{k_\beta}, y^{m_\gamma}$. From (23) we obtain

$$B_{j_\beta l_\beta j_\gamma l_\gamma} = B_{j_\beta l_\beta j_\delta l_\delta} B_{j_\delta l_\delta j_\gamma l_\gamma}.$$

Consequently,

$$B_{j_\beta l_\beta j_\gamma l_\gamma} = B_{j_\beta l_\beta} / B_{j_\gamma l_\gamma}, \tag{24}$$

where $B_{j_\beta l_\beta}$ are functions only of u^{m_β}. From (23) and (24) we find

$$g_{j_\beta l_\beta} / B_{j_\beta l_\beta} \prod_{\sigma \neq \beta} |\varphi_\sigma - \varphi_\beta| = g_{j_\gamma l_\gamma} / B_{j_\gamma l_\gamma} \prod_{\sigma \neq \gamma} |\varphi_\sigma - \varphi_\gamma| = A.$$

Thus, we obtain

$$g_{j_\beta l_\beta} = A B_{j_\beta l_\beta} \prod_{\sigma \neq \beta} |\varphi_\sigma - \varphi_\beta|. \tag{25}$$

From (3) and (25) we find

$$g_{ij} du^i du^j = A \sum_{\alpha=1}^{p} \prod_{\sigma \neq \alpha} |\varphi_\sigma - \varphi_\alpha| \, ds_\alpha^2, \tag{26}$$

$$b_{ij} du^i du^j = A \sum_{\alpha=1}^{p} (\varphi_\alpha F + \Phi) \prod_{\sigma \neq \alpha} |\varphi_\sigma - \varphi_\alpha| \, ds_\alpha^2, \tag{27}$$

where $A > 0$, F, Φ are arbitrary functions; φ_α are functions only of u^α, and $\varphi_\alpha = \text{const}$ if k_α is multiple, $\varphi_\alpha \neq \varphi_\beta$, $\alpha \neq \beta$.

The metric

$$ds^2 = \sum_{\alpha=1}^{p} \prod_{\sigma \neq \alpha} |\varphi_\sigma - \varphi_\alpha| \, ds_\alpha^2$$

is called the Levi-Civita metric ([3]). Consequently, the metric (26) is conformal to the Levi-Civita metric. If, in a space with metric (26), the tensor b_{ij} is chosen according to (27), then it will evidently be generalized recurrent. Thus the following theorem has been proved.

Theorem. *A Riemannian space V_n, $n \geqslant 4$, admits a generalized recurrent symmetric tensor of the second order having* p $(p \geqslant 4)$ *distinct proper values if and only if V_n is conformal to a Levi-Civita space.*

7. Some special cases.

From (10) and (19) we find $\bar{\mu}_i = \partial \ln |F| / \partial u^i$; hence, and from (6), we obtain $\bar{\lambda}_i = \partial \Phi / \partial u^i - \Phi \, \partial \ln |F| / \partial u^i$. From (5) and (7), taking into account the expressions found for $\bar{\lambda}_i$ and $\bar{\mu}_i$, we have

$$2\mu_i = -\partial \ln A / \partial u^i, \qquad 2\lambda_{i_\alpha} = \varphi'_\alpha F + (F\varphi_\alpha + \Phi) \partial \ln A / \partial u^{i_\alpha}.$$

1) A semirecurrent tensor, i.e. $\lambda_i = 0$, $\mu_i = 0$. We find $A = \text{const}$, $\varphi_\alpha = \text{const}$. Consequently, the metric V_n is reducible. In particular, for a recurrent tensor we also have $\bar{\lambda}_i = 0$, since $\Phi = cF$, $c = \text{const}$. Consequently, $k_\alpha = F(\varphi_\alpha + c)$—Datta's theorem ([1]) on the proportionality to one another of the proper values of a recurrent tensor.

2) If g_{ij}, b_{ij} are the first and second fundamental tensors of a hypersurface, then for $\lambda_i = 0$, $\bar{\lambda}_i = 0$, $\bar{\mu}_i = \mu_i$ we obtain the tensor characteristic of a hypersurface ([2]). We find $\Phi = cF$, $c = \text{const}$, $A = \prod_{\sigma=1}^{p} f_\sigma^{-1}$, $f_\alpha = \varphi_\sigma + c$,

$$F = A^{-1/2} = \prod_{\sigma=1}^{p} f_\sigma^{1/2}.$$

Consequently,

$$k_\alpha = f_\alpha \prod_{\sigma=1}^{p} f_\sigma^{1/2}, \qquad g_{i_\alpha j_\alpha} = B_{i_\alpha j_\alpha}(u^{m_\alpha}) \prod_{\sigma \neq \alpha} \left| 1 - \frac{f_\alpha}{f_\sigma} \right|,$$

i.e. we obtain the principal curvatures and the metric of a hyperquadric. At the same time we find

$$\mu_{i_\alpha} = \frac{1}{p+2} \frac{\partial \ln |K|}{\partial u^{i_\alpha}}, \qquad K = \prod_{\sigma=1}^{p} k_\sigma.$$

3) For $\mu_i = 0$, $\bar{\mu}_i = 0$, $\bar{\lambda}_i = 2\lambda_i$, we obtain $F = \text{const}$, $A = \text{const}$,

$$\Phi = F \sum_{\sigma=1}^{p} \varphi_\sigma.$$

Consequently,

$$k_\alpha = f_\alpha + \sum_{\sigma=1}^{p} f_\sigma, \qquad f_\alpha = F\varphi_\alpha, \qquad ds^2 = \sum_{\alpha=1}^{p} \prod_{\sigma \neq \alpha} |f_\sigma - f_\alpha| \, ds_\alpha^2.$$

$$b_{ij}du^i du^j = \sum_{\alpha=1}^{p}\left(f_\alpha + \sum_{\sigma=1}^{p} f_\sigma\right)\prod_{\sigma\neq\alpha}|f_\sigma - f_\alpha|\, ds_\alpha^2, \qquad \lambda_{i_\alpha} = \frac{\partial}{\partial u^{i_\alpha}}\left(\frac{1}{2}\sum_{\sigma=1}^{p} f_\sigma\right).$$

Thus, we obtain the Levi-Civita metric.

4) For $\bar{\lambda}_i = \lambda_i,\ \bar{\mu}_i = \mu_i$ we obtain $A = F^{-2},\ \partial\Phi/\partial u^{i_\alpha} = \frac{1}{2}\varphi'_\alpha F - \varphi_\alpha \partial F/\partial u^{i_\alpha}$, i.e. a generalization of our results ([4]) to the case of multiple curvatures.

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Note: Figure translations are in progress. See original paper for figures.