LINEAR AND NONLINEAR PARABOLIC SYSTEMS IN THE PLANE

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LINEAR AND NONLINEAR PARABOLIC SYSTEMS IN THE PLANE

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In this paper a priori estimates are established for solutions of the first boundary-value problem and the Cauchy problem for linear parabolic equations of the form

$$(-1)^{m+1} D_t u = \sum_{k=0}^{2m} a_k(t,x) D_x^{2m-k} u + f(t,x), \tag{1}$$

$$u = (u^1, \dots, u^N), \quad D_t u = \partial u/\partial t, \quad D_x^i u = \partial^i u/\partial x^i, \quad D_x^0 u \equiv u,$$

m and N are natural numbers, and for nonlinear equations of parabolic type

$$(-1)^{m+1} D_t u = a(t, x, u, D_{xu}, \dots, D_x^{2m} u); \tag{2}$$

in the case $N > 1$, equations (1) and (2) are vector equations, and $a_k(t,x)$ are square matrices of size $N \times N$, $a(t, x, p_0, p_1, \dots, p_{2m})$ is the vector $(a^1(\dots), \dots, a^N(\dots))$ (here and below $(\dots) \equiv (t, x, p_0, p_1, \dots, p_{2m})$, p_i is the vector $(p_i^1, \dots, p_i^N)$, $i = 0, \dots, 2m$).

The estimates obtained are used to prove nonlocal existence theorems for solutions of the problems under consideration.

Among equations (1) and (2), scalar equations of second order occupy a special place, i.e. the case $m = N = 1$; in this case the maximum principle is valid for solutions. The study of such equations, based on the maximum principle, was carried out in the works (^{1,2}), and the fundamental role here is played by an a priori estimate of the derivative $u_x(t,x)$, established under minimal assumptions on the coefficients of the equations and on the solutions. An analogous role in the general case considered in the present paper is played by a priori estimates in L_2 of the higher derivatives D_{tu} and $D_x^{2m} u$ for a solution $u(t,x)$ of equation (1), valid without any assumptions on the continuity of the coefficients of the equation. In the case $m = N = 1$, such estimates were obtained and applied to the study of certain quasilinear equations in the works ($^{3-5}$).

Let Q_T be the rectangle $\{(t,x) : 0 \le t \le T,\ |x| \le l\}$; by Π_T we shall denote the strip $\{(t,x) : 0 \le t \le T,\ -\infty < x < +\infty\}$.

The basic assumption concerning equations (1) and (2) is the condition of strong parabolicity: for any real vector $\xi = (\xi^1, \dots, \xi^N)$ and any $(t,x) \in Q_T$ (or $(t,x) \in \Pi_T$, if the Cauchy problem is considered) the inequality

$$(\xi, a_0(t,x)\xi) \ge \lambda|\xi|^2, \qquad \lambda = \text{const} > 0 \tag{3}$$

holds (here (ξ, η) is the scalar product of the vectors ξ and η in N-dimensional Euclidean space, $|\xi| = (\xi,\xi)^{1/2}$); the condition of strong parabolicity for system (2) means that inequality (3) is satisfied by the matrix

$$A_0(\dots) \equiv (a^i_{p^j_{2m}}(\dots)), \qquad 1 \le i,j \le N,$$

for all $(t,x) \in Q_T$ (or $(t,x) \in \Pi_T$) and all $p_0, p_1, \dots, p_{2m}$.

Denote by $W_q^{1,2m}(Q_T)$, $q \geqslant 2$, the Banach space of vector-functions $u(t,x)$ with norm

$$\|u\|_q^{1,2m} = \left[\iint_{Q_T} (|u|^q + |D_{tu}|^q + |D_x^{2m} u|^q)\, dx\, dt\right]^{1/q} \tag{4}$$

(the space $W_q^{1,2m}(\Pi_T)$ is defined analogously). From embedding theorems (see, for example, (^{6})) it is known that every function $u \in W_q^{1,2m}(Q_T)$ is continuous in Hölder sense in Q_T, together with the derivatives $D_x^i u$, $i = 1, \dots, m-1$. Denote by $\dot{H}_q(Q_T)$ the subspace of functions $W_q^{1,2m}(Q_T)$ satisfying the conditions

$$u|_{t=0} = 0, \qquad D_{xu}|_{x=\pm l} = 0, \qquad i = 1, \dots, m-1 \tag{5}$$

(by $H_q(\Pi_T)$ we shall denote the subspace of functions in $W_q^{1,2m}(\Pi_T)$ satisfying the first of conditions (5)).

Let us note that the theorems formulated in the present note admit some generalizations which do not require the use of other methods of proof. In particular, analogous theorems hold for strongly parabolic systems with complex coefficients; a number of results can be transferred to parabolic equations with many independent variables having a special structure.

1. Linear parabolic equations. We shall assume that all elements of the matrices $a_k(t,x)$ are measurable, and that their moduli for $(t,x) \in Q_T$ (or $(t,x) \in \Pi_T$) do not exceed the number M.

Theorem 1. *If the function* $u \in \dot{H}_2(Q_T)$ *(or* $u \in \dot{H}_2(\Pi_T)$*) is a solution of problem (1), (5) (or a solution of the Cauchy problem for equation (1) with zero initial condition), then*

$$\operatorname*{vrai\,max}_{0\leqslant t\leqslant T} \left(\int [D_x^m u(t,x)]^2\,dx\right)^{1/2} + \|u\|_2^{1,2m} \leqslant C\|f\|_{L_2}, \tag{6}$$

where $C = C(m, N, \lambda, M, \min(1,l), T)$.

To prove Theorem 1, equation (1) is scalarly multiplied by the vector $e^{\theta t}D_x^{2m}u$, with a sufficiently large number $\theta < 0$, and then estimate (6) is derived by integration by parts and the application of certain elementary inequalities.

Theorem 2. *For every function* $f \in L_2(Q_T)$ *(or* $f \in L_2(\Pi_T)$*) there exists a unique solution* $u(t,x) \in \dot{H}_2(Q_T)$ *(or* $u(t,x) \in \dot{H}_2(\Pi_T)$*) of problem (1), (5) (or of the Cauchy problem with zero initial condition).*

The corresponding results for parabolic equations with continuous coefficients were obtained in papers ($^{7-9}$); moreover, for such equations analogous theorems are also valid in the spaces $W_q^{1,2m}$ for any $q \geqslant 2$; in particular, the estimate

$$\|u\|_q^{1,2m} \leqslant C_q\|f\|_{L_q} \tag{7}$$

holds (for the case $q = \infty$, see paper (^{10})). The example of the function

$$u(t,x) \equiv [x^{2m} + (-1)^{m+1}t]\,(x^{2m} - t)^{-\alpha}, \qquad \alpha > 0,$$

which satisfies in the rectangle $\{(t,x) : -1 \leqslant t \leqslant 0,\ |x| \leqslant 1\}$ the parabolic equation $(-1)^{m+1}D_{tu} = a(t,x)D_x^{2m}u$ for a sufficiently small number $\alpha > 0$ (moreover, $(2m)! - 1 \leqslant a^{-1}(t,x) \leqslant (2m)! + 1$), shows that for equations of the form (1) with discontinuous coefficients at the highest derivatives, estimate (7) is false for large q. However, the following is valid.

Theorem 3. *Suppose that, in addition to the above-stated conditions on the coefficients of equation (1), the matrix* $a_0(t,x)$ *is symmetric and let*

$u(t,x) \in \dot{H}_2(Q_T)$, $f(t,x) \in L_q(Q_T)$, $q \geq 2$. Then there exists a number q_0, independent of u, f, and q, such that for $2 \leq q \leq q_0$ the estimate (7) is valid.

In the proof of Theorem 3, some elements are used from the methodology of the works of I. N. Vekua on the theory of elliptic equations (see (^{11}), Ch. 4, § 9), connected with the application of M. Riesz’ s convexity theorem (see (^{12}), Ch. 6, § 10).

Remark 1. Analogs of Theorems 1 and 2 can also be established for certain degenerating equations. For example, in the case $N = 1$ and $a_k \equiv 0$ for $k =$

$1, \dots, 2m$ (one equation containing only the highest derivatives), for the solution of problem (1), (5) the estimate holds

$$\operatorname{vrai}\max_{0\le t\le T}\int_{-l}^{l}[D_x^m u(t,x)]^2\,dx+\iint_{Q_T}\left[\frac{1}{a_0}(D_{tu})^2+a_0(D_x^{2m}u)^2\right]dx\,dt\le\iint_{Q_T}\frac{1}{a_0}f^2\,dx\,dt.$$

2. Nonlinear parabolic equations. We present here theorems on the solvability of the first boundary-value problem for nonlinear equations; the formulations of theorems on the solvability of the Cauchy problem are analogous. First we consider the case of a quasilinear equation

$$(-1)^{m+1}D_{tu}=\sum_{k=0}^{2m}a_k(t,x,u,D_{xu},\dots,D_x^{2m-1}u)D_x^{2m-k}u+f(t,x,u,D_{xu},\dots,D_x^{2m-1}u) \tag{8}$$

(in this case $A_0(\dots)\equiv a_0(\dots)$, $(\dots)\equiv(t,x,p_0,\dots,p_{2m-1})$). We shall assume that all elements of the matrices $a_k(\dots)$ and the components of the vector $f(\dots)$ are locally bounded, measurable, and continuous in $p_0,\dots,p_{2m-1}$. Let the following conditions be satisfied:

1) for any real vector ξ

$$|a_k'(\dots)\xi|\le\operatorname{const}(a_0(\dots)\xi,\xi)^{1/2},\qquad k=1,\dots,2m; \tag{9}$$

$$|(f(\dots),\xi)|\le\operatorname{const}(a_0(\dots)\xi,\xi)^{1/2}\left[\sum_{i=0}^{2m-1}|p_i|+\varphi(t,x)\right],\qquad\varphi\in L_2(Q_T) \tag{10}$$

(a_k' is the matrix transposed to a_k);

2) the elements of the matrices $a_k(\dots)$ are bounded in domains of the form

$$D_M=\{(t,x)\in Q_T;\ |p_i|\le M,\ i=0,\dots,m-1;\ -\infty<p_i<+\infty,\ i=m,\dots,2m\}.$$

Theorem 4. Problem (8), (5) has at least one solution $u(t,x)\in\dot{H}_2(Q_T)$. Such a solution is unique if the elements of the matrices $a_k(\dots)$ and the components of the vector $f(\dots)$ satisfy the Lipschitz condition in $p_0,\dots,p_{2m-1}$ in domains of the form D_M.

For the proof of Theorem 4 one applies Schauder's fixed-point theorem, Theorem 2, a priori estimates analogous to (6), and also some approximation considerations.

Theorem 5. Suppose that, in addition to all the requirements indicated above, the elements of the matrix $a_0(\dots)$ depend only on $(t,x,p_0,\dots,p_{m-1})$, the elements of all matrices $a_k(\dots)$ and the components of the vector $f(\dots)$ satisfy a Hölder condition in t and x when the remaining arguments are uniformly bounded. Suppose that the compatibility condition $f(0,\pm l,0,\dots,0)=0$ is satisfied. Then problem (8), (5) has in Q_T a unique solution possessing in Q_T continuous Hölder derivatives entering equation (8).

This assertion is also valid in the case of somewhat stronger growth of the elements of the vector $f(\dots)$ with respect to $r = |p_0| + \dots + |p_{m-1}|$; for example, the term $(r + |p_m|)\sqrt{\ln(1 + r)}$ may be added to the square bracket on the right-hand side of inequality (10).

Example 1. The conditions of Theorem 5 are satisfied, in particular, by the equation $(N = 1)$

$$(-1)^{m+1} D_{tu} = a(t, x, u, \dots, D_x^{m-1} u) D_x^{2m} u + f(t, x, u, \dots, D_x^{m-1} u),$$

where $a(\dots)$ and $f(\dots)$ are smooth functions, $a(\dots) \geq \lambda = \text{const} > 0$,

$$|f(\dots)| \leq \text{const}\, \sqrt{a(\dots)} (r + 1) \ln^{1/2}(r + 2), \quad f(0, \pm l, 0, \dots, 0) = 0.$$

The following theorem on the existence of a classical solution of the first boundary-value problem for the nonlinear equation (2) is valid.

Theorem 6. *Let the components of the vector* $a(\dots)$ *have continuous derivatives with respect to all arguments, and let the inequalities (9) be satisfied, where* $a_k(\dots)$ *are the matrices* $A_k(\dots) = (a^{pj}_{2m-k}(\dots))$, $k = 0, \dots, 2m$. *Suppose that for* $(\dots) \in D_M$ *and any real vector* ξ *the estimate*

$$|(a_t(\dots), \xi)| \leq \text{const}(A_0(\dots)\xi, \xi)^{1/2} \left(\sum_{i=m}^{2m} |p_i| + 1 \right).$$

holds. Assume that $D_x^m a(0, x, 0, \dots, 0) \in L_2(-l, l)$ *and that the compatibility condition* $a(0, \pm l, 0, \dots, 0) = 0$ *is fulfilled. Then there exists a unique solution* $u(t, x)$ *of problem (2), (5), possessing in* Q_T *continuous, Hölder-continuous derivatives entering equation (2); moreover* $D_{tu} \in W_2^{1,2m}(Q_T)$.

Example 2. All the conditions of Theorem 6 are satisfied, in particular, by the equation

$$(-1)^{m+1} D_{tu} = F(D_x^{2m} u) + f(t, x, u, \dots, D_x^{2m-1} u),$$

where $F(p_{2m})$ and $f(\dots)$ are smooth functions, $F'(p_{2m}) \geq \lambda = \text{const} > 0$, $F(0) + f(0, x, 0, \dots, 0) = 0$, and the function $f(\dots)$ satisfies the Lipschitz condition in all variables.

Remark 2. In accordance with Theorem 3, one can prove an analogue of Theorem 4 also for the case of the space $\mathring{H}_q(Q_T)$ $(2 \leq q < q_0)$, under the condition that the elements of the matrices a_k are bounded and under the assumption that $\varphi(t, x) \in L_q(Q_T)$ (see (10)).

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