ON TRIGONOMETRIC SUMS

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Abstract

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In the paper [2] Hua Loo-keng proved:

If $n \geqslant 2$, $(a, q) = 1$, $1 \leqslant P \leqslant q$, then

$$S(a, P; q) = \sum_{1 \leqslant x \leqslant P} \exp 2\pi i \frac{ax^n}{q} = Pq^{-1} S(a, q; q) + O\left(q^{1/2+\varepsilon}\right), \tag{1}$$

where $\varepsilon > 0$ is arbitrary, and the constant in O depends only on n and ε.

This asymptotic formula for a trigonometric sum; formula (1) is nontrivial if the complete sum $S(a, q; q)$ is "large," i.e. its modulus is greater than $q^{1/2+\varepsilon}$; the latter occurs only when q contains "many" prime factors to a "high" power. We note that, even in the best case, (1) is an asymptotic formula for $q^{1/2+\varepsilon} \ll P \leqslant q$, i.e. if the original sum $S(a, P; q)$ is "sufficiently long."In [3, 4] the author obtained an asymptotic formula for "short" sums $S(a, P; q)$, if q is a "high" power of a prime number. However, Theorem 1 in [3] and Lemma 4 in [4] were formulated incorrectly. In the present article new asymptotic formulas will be obtained when q is a power of a prime number.

Notation and conditions on the parameters used.
$k, l, m, n, t, P, P_1, a, a_1, \dots, a_n, b$ are integers; $n \geqslant 20$; r is a real number, with $1 \leqslant r \leqslant 0.1n$; $t \geqslant 4rn$; $C, C_1, \dots, \gamma, \gamma_1, \dots$ are positive absolute constants; $\varepsilon > 0$ is an arbitrarily small number; p is a prime number, $p > \exp cn \ln n$, $p/\ln p > t$; $q = p^t$; $P = q^{1/r}$; $(a, p) = 1$; $\delta_n(x) = 1$ or 0, according as $x \equiv 0 \pmod n$ or $x \not\equiv 0 \pmod n$; the constants in the signs O and $\ll$ depend only on n; as above,

$$S(a, P; q) = \sum_{1 \leqslant x \leqslant P} \exp 2\pi i \frac{ax^n}{q}.$$

Theorem 1. There exist $\varphi(q)(1 - 2q^{-\varepsilon})$ values of a such that, for every P, the asymptotic formula holds

$$S(a, P; q) = Pq^{-1} S(a, q; q) + O\left(\sqrt{Pq^{\varepsilon} \ln^3 q} + P^{3/4}\right).$$

Theorem 2. For every a the equality holds:

$$S(a, P; q) = Pq^{-1} S(a, q; q) + O\left(P^{1-\gamma/r^2}\right).$$

Proof of Theorem 1. Let a be arbitrary, $(a, q) = 1$.

1. Separation of the principal term. Define integers s and m from the conditions $2^s \leqslant P < 2^{s+1}$, $p^{m-1} \leqslant 2^{0.5s} < p^m$. Then $s = [\frac{t}{r}\log p]$, $m - 1 = (0.5s/\log p)$; $m - 1 < t/2r \leqslant m + 1/2t$; moreover, $p^m \ll P^{3/4}$. Therefore we have

$$S(a, P; q) = \sum_{1 \leqslant x \leqslant P_1 p^m} \exp 2\pi i \, \frac{ax^n}{q} + O\left(P^{3/4}\right) =$$

$$= \sum_{0 \leqslant b \leqslant P_1 - 1} \sum_{1 \leqslant x \leqslant p^m} \exp 2\pi i \, \frac{a(x + bp^m)^n}{q} + O\left(P^{3/4}\right). \tag{2}$$

Let us now consider the inner sum

$$S^*(a) = \sum_{1 \leq x \leq p^m} \exp 2\pi i \, \frac{a(x + bp^m)^n}{q}.$$

Let $t = t_1 n + t_2$, $0 \leq t_2 \leq n - 1$, $t_1 = [t/n]$; $r_m = t/m$; then $r_m \leq 2.001r$ and $r_m \geq t/(t/2r + 1) \geq 1$. We split the sum $S^*(a)$ into $t_1 - \delta_n(t) + 1$ sums, grouping together the terms with x not divisible by p, divisible by p but not by p^2, divisible by p^2 but not by p^3, etc. From the definition of r_m and the conditions on t and r_m we obtain: $r_m = t/m \leq 0.2001n$, $m \geq 5t/n - 1/2t > t_1 + 1$; $n(t_1 - \delta_n(t) + 1) \geq t$. Consequently,

$$S^*(a) = p^{m - t_1 + \delta_n(t) - 1} + \sum_{0 \leq \nu \leq t_1 - \delta_n(t)} S^*_\nu(a), \tag{3}$$

where

$$S^*_\nu(a) = \sum_{\substack{1 \leq x \leq p^{m-\nu} \\ (x,p)=1}} \exp 2\pi i \, \frac{a(x + bp^{m-\nu})^n}{p^{t-\nu n}}, \qquad \nu = 0, 1, \ldots, t_1 - \delta_n(t).$$

Now we split the sum over ν in (3) into two parts:

a)
$$0 \leq \nu \leq \nu_0 = \left[\frac{t}{n-1}\left(1 - \frac{1}{r_m}\right)\right];$$

b)

$$\nu_0 + 1 \le \nu \le t_1 - \delta_n(t).$$

Each of the intervals a) and b) is nonempty. Indeed, case a) is trivial. In case b) we have:

$$\nu_0 + 1 \le \frac{t}{n-1}\left(1 - \frac{1}{r_m}\right) + 1 \le \frac{t}{n}\left(1 + \frac{1}{n-1}\times\right)$$
$$\times\left(1 - \frac{1}{2.001r}\right) < \frac{t}{n} - 1 \le t_1 - \delta_n(t).$$

Consider case b). Then $m - \nu \ge t - \nu n$; indeed, this follows from the inequality

$$\nu \ge (t-m)/(n-1) = t(1-1/r_m)/(n-1).$$

Consequently, $S_\nu^*(a)$ is a complete sum, and

For any $l \ge 2$ we have the equality:

$$S_\nu^*(a) = \sum_{\substack{1\le x\le p^{t-\nu n}\\(x,p)=1}} \exp 2\pi i\,\frac{ax^n}{p^{t-\nu n}}.$$

$$\sum_{\substack{1\le x\le p^{l}\\(x,p)=1}} \exp \frac{ax^n}{p^l} = 0$$

(see, for example, (1), p. 270). Consequently, only $S_{t_1}^*(a)$ can be different from zero, and this will occur only when $t_2 = 1$, i.e.

$$S_{t_1}^*(a) = \delta_n(t-1)p^{m-t_1-1}\sum_{x=1}^{p-1} \exp 2\pi i\,\frac{ax^n}{p} = \delta_n(t-1)p^{m-t_1-1}(S(a,p;p)-1).$$

Thus,

$$S^*(a) = p^{m-t_1+\delta_n(t)-1} + \delta_n(t-1)p^{m-t_1-1}(S(a,p;p)-1)+$$
$$+\sum_{0\le\nu\le\nu_0} S_\nu^*(a) = p^m q^{-1} S(a,q;q) + \sum_{0\le\nu\le\nu_0} S_\nu^*(a),$$

which also follows from (1), p. 270. From the last formula, the definition of $S_\nu^*(a)$, and (2) we find

$$S(a,P;q) = Pq^{-1}S(a,q;q) + R(a,P;q) + O(P^{3/4}), \tag{4}$$

where

$$R(a,P;q) = \sideset{}{'}\sum_{1\le x\le P} \exp 2\pi i\,\frac{ax^n}{q},$$

and the prime on the sum means that the summation is over $x \not\equiv 0 \pmod{p^{\nu_0+1}}$.

2. Estimate of the remainder. The quantity $|R(a,P;q)|$ depends essentially on P. We shall now pass to a larger quantity which will depend only

on n, a, and q. Note that ν_0 depends only on n, t, and s. We have

$$R(a,P;q) = 2^{-s-1} \sideset{}{'}\sum_{1\le x\le 2^{s+1}} \exp 2\pi i\,\frac{ax^n}{q} \sum_{0\le b\le 2^{s+1}} \sum_{y\le P} \exp 2\pi i\,\frac{b(x-y)}{2^{s+1}},$$

where the prime, as before, means that $x \not\equiv 0 \pmod{p^{\nu_0+1}}$. Separating the term with $b=0$, summing over y, and passing to inequalities, we obtain

$$|R(a,P;q)| \ll \sum_{0\le b\le 2^{s+1}} \frac{1}{b+1} \left| \sideset{}{'}\sum_{1\le x\le 2^{s+1}} \exp 2\pi i \left(\frac{ax^n}{p^t} - \frac{bx}{2^{s+1}}\right)\right|.$$

Consequently,

$$|R(a,P;q)|^2 \ll \log q \sum_{0\le b\le 2^{s+1}} \frac{1}{b+1} \left| \sideset{}{'}\sum_{1\le x\le 2^{s+1}} \exp 2\pi i \left(\frac{ax^n}{p^t} - \frac{bx}{2^{s+1}}\right)\right|^2,$$

$$P^{-1}|R(a,P;q)|^2 \ll 2^{-s-1}\log q \sum_{0\le b\le 2^{s+1}} \frac{1}{b+1} \left| \sideset{}{'}\sum_{1\le x\le 2^{s+1}} \exp 2\pi i \left(\frac{ax^n}{p^t} - \frac{bx}{2^{s+1}}\right)\right|^2$$

$$\ll \log q \sum_{0\le s\le \log q} 2^{-s-1} \sum_{0\le b\le 2^{s+1}} \frac{1}{b+1} \left| \sideset{}{'}\sum_{1\le x\le 2^{s+1}} \exp 2\pi i \left(\frac{ax^n}{p^t} - \frac{bx}{2^{s+1}}\right)\right|^2 = \log q \cdot \Phi,$$

where $\Phi = \Phi(a,q) = \Phi(n,a,q) \ge 0$ and depends only on n, a, and q.

Arrange $\Phi(a,q)$ in increasing order: $\Phi(a_1,q) \le \Phi(a_2,q) \le \cdots \le \Phi(a_\varkappa,q)$, $\varkappa = \varphi(q)$. For $1 \le \omega < \varkappa$ we have

$$\Phi(a_\omega,q) \ll \frac{1}{\varkappa-\omega+1} \sum_{\omega\le m\le\varkappa} \Phi(a_m,q) \ll \frac{1}{\varkappa-\omega+1} \sum_{1\le m\le\varkappa} \Phi(a_m,q) \ll$$

$$\ll \frac{1}{\varkappa - \omega + 1} \sum_{0 \le s \le \log q} 2^{-s-1} \sum_{0 \le b \le 2^{s+1}} \frac{1}{b+1}\, qT_s,$$

where T_s is the number of solutions of the congruence $x^n \equiv y^n \pmod{p^t}$, $1 \le x, y \le 2^{s+1}$, $x \not\equiv 0 \pmod{p^{\nu_0+1}}$, $y \not\equiv 0 \pmod{p^{\nu_0+1}}$. Since $2^{s+1}p^{-\nu_0} \ll p^{t-\nu_0 n}$, it follows that

$$T_s \ll \sum_{0 \le \nu \le \nu_0} 2^{s+1} p^{-\nu} \ll 2^{s+1}, \qquad \Phi(a_\omega, q) \ll \frac{q \log^2 q}{\varkappa - \omega + 1}.$$

If now we take $\omega = \varkappa - q^{1-\varepsilon}$, then

$$\Phi(a_\omega, q) \ll q^\varepsilon \log^2 q, \qquad P^{-1}|R(a_\nu, P; q)|^2 \ll q^\varepsilon \log^3 q$$

for $\nu = 1, 2, \dots, \varphi(q) - q^{1-\varepsilon}$. From this and from (4) we obtain the assertion of Theorem 1.

For the proof of Theorem 2 we shall need the following

Lemma. Let $q_1 = p^k$, $n \le k \le t$, $1 \le u \le 0.5n$; $P_1^u = q_1$, $(a_1, p) = (a_2, p) = \dots = (a_n, p) = 1$,

$$S = \sum_{1 \le x \le P_1} \exp 2\pi i\, \frac{a_1 x + a_2 p x^2 + \dots + a_{np}^{n-1} x^n}{q_1}.$$

Then

$$|S| \le c_1 P_1^{1-\gamma_1/u^2}.$$

This is one of the variants of the theorem of [5], p. 239.

Proof of Theorem 2. Estimate $|R(a, P; q)|$ in (4) (see the proof of Theorem 1). We have

$$R(a, P; q) = \sum_{1 \le x < P}{}' \exp 2\pi i\, \frac{ax^n}{q} = \sum_{0 \le h < P_1 - 1} \sum_{0 \le \nu \le \nu_0} S_\nu^*(a) + O(P^{3/4}).$$

In the sum $S_\nu^*(a)$ make the change of summation variable of the form $x = py + z$, $1 \le z \le p-1$, $0 \le y \le p^{m-\nu-1} - 1$; then $ax^n = a(py+z)^n = a_0 + a_1 py + \dots + a_{np}^{ny^n}$, where $(a_0, p) = \dots = (a_n, p) = 1$ for every z, $1 \le z \le p-1$. Further,

$$|S_\nu^*(a)| \le \sum_{1\le z\le p-1} \left| \sum_{0\le y\le p^{m-\nu-1}} \exp 2\pi i \, \frac{a_1 y + a_2 p y^2 + \cdots + a_{np}^{n-1} y^n}{p^{t-\nu n-1}} \right|.$$

The estimate of the lemma is applicable to the last sum. Let us verify that the conditions of the lemma are satisfied:

$k = t - \nu n - 1; \quad k \le t; \quad k \ge t - \nu_0 n - 1 \ge t - 1 -$

$$-t\left(1+\frac{1}{n-1}\right)\left(1-\frac{1}{r_m}\right) \ge t-1-t\left(1+\frac{1}{n-1}\right)\left(1-\frac{1}{2.001r}\right) \ge n;$$

let $u = (t-\nu n-1)/(m-\nu-1)$, then $u \ge 1$, since $\nu \le (t-m)/(n-1) = t(1-1/r_m)/(n-1)$; moreover, $u = r_m(t-\nu n+1)/(t-\nu r_m - r_m) \le 2r_m$, which follows from the inequality $r_m \le \frac{1}{2}(\nu n+t+1)/(\nu+1) = \frac{1}{2}(n+(t+1-n)/(\nu+1))$. Thus, always $1 \le u \le 2r_m \le 4.002\,r < 0.5\,n$. Applying the estimate of the lemma, we obtain:

$$|S_\nu^*(a)| \ll p \cdot p^{(m-\nu-1)(1-\gamma_2/u^2)} = p^{(m-\nu)(1-\gamma_3/u^2)} = p^{(m-\nu)(1-\gamma_4/r_m^2)}.$$

Thus,

$$|R(a,P;q)| \ll P_1 \sum_{0\le\nu\le\nu_0} p^{(m-\nu)(1-\gamma_1/r_m^2)} + P^{3/4} \ll$$

$$\ll P_1 p^{m(1-\gamma_1/r_m^2)} + P^{3/4} \ll P^{1-\gamma/r^2}.$$

From (4) and the last estimate, the assertion of Theorem 2 follows.

Remark 1. Theorem 1 can be proved for a wider interval of variation of the values of P, namely $1 \le r \le 0.5n$.

Remark 2. Hua Lo-keng' s theorem and Theorems 1 and 2 establish the asymptotics of the corresponding trigonometric sums for $1 \le r \le 2$, $1 \le r \le 0.1n$, and $1 \le r \le c\sqrt{n}$, respectively.

Remark 3. Theorems 1 and 2 can be generalized to a broader class of trigonometric sums.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.