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Abstract

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ON INTEGRAL INVARIANTS IN SYSTEMS OF HYDRODYNAMIC TYPE

The equations of hydrodynamics of an ideal incompressible fluid (i.i.f.) have the characteristic property that they are nonlinear in the velocity field, this nonlinearity being of second degree, and, moreover, there exists a quadratic integral of the motion—the energy. These equations may be written in the form

$$\partial \mathbf{v}/\partial t = \operatorname{rot}^{-1}\{(\operatorname{rot}\mathbf{v}\nabla)\mathbf{v} - (\mathbf{v}\nabla)\operatorname{rot}\mathbf{v}\}, \tag{1}$$

where the state of the system is characterized by a solenoidal velocity field $\mathbf{v}(\mathbf{r})$ ($\operatorname{div}\mathbf{v} = 0$), satisfying the boundary conditions $v_n = 0$ on fixed walls. The region occupied by the fluid is assumed finite, and the discussion is carried out in a coordinate system in which the center of gravity of the system is at rest. The symbol rot^{-1} denotes the operator of reconstructing the velocity field from the vorticity field with allowance for the boundary conditions. The classical Euler equations for the dynamics of a rigid body possess the same general properties:

$$I_1\dot\omega_1 = (I_2 - I_3)\omega_2\omega_3, \quad I_2\dot\omega_2 = (I_3 - I_1)\omega_3\omega_1, \quad I_3\dot\omega_3 = (I_1 - I_2)\omega_1\omega_2 \tag{2}$$

with the energy integral

$$2E = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2. \tag{3}$$

We use the notation adopted in ([1]).

The analogy between the equations of i.i.f. and the dynamics of a rigid body (r.b.) has deep group-theoretic roots, as was shown not long ago by V. I. Arnold ([2]). In the present work a definition is given of a broad class of dynamical systems, including rigid-body dynamics and finite-dimensional analogues of i.i.f., for the purpose of considering the question of integral invariants, which is important for the statistical theory of turbulence.

We shall call a mechanical system S_n, whose state is determined by a set of quantities $u^1, u^2, \dots, u^n$ (the vector u^i), a system of hydrodynamic type (h.t.s.) if the three fundamental conditions formulated below are satisfied.

1. The equations of motion are quadratically nonlinear and may be written, with the aid of a dynamical tensor of third rank Γ^i_{jk}, in the form

$$\dot{u}^i = \Gamma^i_{\lambda\mu} u^\lambda u^\mu, \tag{4}$$

where it is understood that summation is performed over identical indices denoted by Greek letters. The dynamical tensor may always be assumed symmetric in the lower indices,

$$\Gamma^i_{jk} = \Gamma^i_{kj}. \tag{5}$$

2. During the motion of the system the phase volume is conserved, which, by Liouville's theorem (see, for example, (^{3})), leads to the following regularity condition:

$$\partial \dot{u}^\lambda / \partial u^\lambda = 0, \tag{6}$$

whence follows the vanishing of the covariant vector

$$V_j = \Gamma^\lambda_{j\lambda} = 0. \tag{7}$$

3. There exists an energy integral—a positive definite quadratic form of the variables

$$2E = g_{\lambda\mu} v^\lambda v^\mu, \tag{8}$$

defined by a nonsingular symmetric matrix g_{ik} and such that

$$dE/dt = 0 \tag{9}$$

by virtue of the equations of motion (4).

The existence of an energy integral imposes on the dynamical tensor Γ^i_{jk} the conditions

$$g_{i\lambda}\Gamma^\lambda_{jk} + g_{j\lambda}\Gamma^\lambda_{ki} + g_{k\lambda}\Gamma^\lambda_{ij} = 0, \tag{10}$$

which are easily proved if one takes into account the symmetry conditions (5). If one introduces the convention of lowering indices with the aid of the energy tensor, then the conservativity condition of the system (10) can be written in the form of the cyclic relation

$$\Gamma_{i,jk} + \Gamma_{j,ki} + \Gamma_{k,ij} = 0, \tag{11}$$

which is convenient to use in a “Cartesian” system of coordinates, when $g_{ik} = \delta_{ik}$.

Let us note that the regularity condition (7) is fulfilled in the case of an infinite-dimensional dynamical system of a G.H.T., for which the phase space is a separable Hilbert space (the metric is introduced with the aid of the energy integral), and the system can be approximated by an infinite sequence of dynamical systems of the type indicated above. For hydrodynamic fields periodic in space, fulfillment of condition (5) is easily proved by the fact that, in the Fourier representation, any term Γ^{λ}_{ij} entering the sum $\Gamma^{\lambda}_{i\lambda}$ turns out to be equal to zero. This circumstance was pointed out by Lee (4). Let us note that finite-dimensional approximations of the equations of hydrodynamics used in the numerical forecasting of meteorological fields (see, for example, (5)) also belong to the class of S.G.T. The regularity condition is then not difficult to satisfy, and in the commonly used difference schemes it holds (in the limit, for an infinitely small time step).

Counting the number of conditions satisfied by the dynamical tensor Γ^{i}_{jk} shows that the number of independent components of this tensor (for a given energy tensor) is

$$N = n(n+2)(n-2)/3, \tag{12}$$

which gives, for $n = 3$ (the simplest case of an S.G.T.), $N = 5$. The question of the equivalence of S.G.T. and the corresponding invariants naturally arises. It can be proved that any S.G.T. with $n = 3$ is equivalent to the system of Euler equations for the dynamics of a rigid body. With the aid of an orthogonal transformation leaving invariant the energy tensor $g_{ik} = \delta_{ik}$, the dynamical tensor is brought to canonical form, in which $\Gamma_{123} = p$, $\Gamma_{231} = q$, $\Gamma_{312} = r$, while all the remaining components vanish; moreover the quantities p, q, r are connected by a single relation

$$p + q + r = 0. \tag{13}$$

The corresponding canonical form of the equations of motion has the form

$$du_1/dt = pu_2u_3, \qquad du_2/dt = qu_3u_1, \qquad du_3/dt = ru_1u_2. \tag{14}$$

When condition (13) is fulfilled, this system, in addition to the energy integral $2E = u_1^2 + u_2^2 + u_3^2$, has also a second quadratic integral, connected with conservation of the square of the angular momentum *

$$I = (q-r)u_1^2 + (r-p)u_2^2 + (p-q)u_3^2. \tag{15}$$

* An elegant proof of the possibility of reducing any S.G.T. with $n = 3$ to the form (14), based on consideration of the matrix $A^i_k = \Gamma_{\lambda,k\mu}e^{i\lambda\mu}$ (where the

symbol e^{ipq} denotes the fundamental tensor, antisymmetric in all three indices), was proposed by L. A. Dikii.

In the general case, in finding invariant characteristics of an h.t.s. one can use the symmetric tensor c_{ik}, formed from Γ^i_{jk} by multiplication and double contraction

$$c_{ik} = \Gamma^{\lambda}_{i\mu}\Gamma^{\mu}_{k\lambda}. \tag{16}$$

The existence of c_{ik} together with g_{ik} makes it possible to construct a certain natural coordinate system in which both of these matrices are brought to diagonal form.

In the statistical description of an h.t.s. it is natural to use the general methods adopted in statistical mechanics, introducing the phase density $\rho(u^i, t)$ and its logarithm $\sigma = \log(1/\rho)$. By virtue of condition (6), the latter characteristic satisfies the well-known Liouville equation

$$\partial\sigma/\partial t + \dot{u}_i \partial\sigma/\partial u_i = 0, \tag{17}$$

and a stationary distribution ρ (respectively σ) is an integral of the motion. For any h.t.s. there exists the stationary Boltzmann distribution

$$\sigma_{\Theta}(u_i) = g_{ik}u^i u^k/\Theta + \text{const.}, \tag{18}$$

where Θ is an analogue of temperature.

For the theory of turbulence, of interest are only nontrivial stationary distributions distinct from the Boltzmann distribution (which plays the role of "white noise"). In an h.t.s., nontrivial stationary distributions of Gaussian type, for which

$$\sigma(u_i) = \sigma_{\lambda\mu}u^{\lambda}u^{\mu}, \tag{19}$$

exist only in the case where there is a quadratic integral of the motion distinct from the energy. The simplest example S_3 shows that such cases are possible, although they are apparently the exception rather than the general rule. A second example of the same kind concerns two-dimensional motions of an ideal incompressible fluid. Consider such a motion in a two-dimensional domain R, bounded by the contour $\mathcal{L}$. The state of the system here is determined by the stream function $\psi(x, y, t)$ with the boundary condition $\psi_L = 0$, and the equations of motion have the form

$$\partial\psi/\partial t = \Delta^{-1}[\Delta\psi, \psi], \tag{20}$$

where Δ^{-1} is the Green operator for the Laplace equation, and

$$[\chi, \psi] = \frac{\partial \chi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \chi}{\partial y}\frac{\partial \psi}{\partial x}.$$

There exist two quadratic integrals—the energy integral and the integral of the square of the vorticity:

$$2E = \iint_R (\operatorname{grad} \psi)^2 \, dx \, dy, \tag{21}$$

$$I = \iint_R (\Delta \psi)^2 \, dx \, dy. \tag{22}$$

In this case there exists a family of stationary Gaussian distributions depending on two parameters. In explicit form it is convenient to describe it with the aid of the second moments for the spectral components ψ_i of the field $\psi(x, y)$, specified by the formula

$$\psi(x, y, t) = \psi^\lambda(t)\chi_\lambda(x, y),$$

where $\chi_i(x, y)$ is a solution of the equation

$$\Delta \chi_i = -k^2_{(i)}\chi_i, \qquad [\chi_i]_L = 0, \qquad \iint_R \chi_i^2 \, dx \, dy = 1. \tag{23}$$

For the characteristics of the field ψ, averaged with the aid of the phase density (the averaging of the quantity $T(u_i)$ is denoted by the symbol $\langle T \rangle$), it is not difficult to obtain the following expression:

$$\langle \psi^i \rangle = 0,$$

$$\langle \psi^i \psi^j \rangle = B_\psi^{ij} = \begin{cases} 0, & \text{for } i \neq j, \\ E/k^2 = C^2/k^2 \left(1 + L_0^2 k^2\right), & \text{for } i = j, \end{cases} \tag{24}$$

where C and L_0 are arbitrary constants; the first characterizes the intensity, and the second the scale of the corresponding "two-dimensional turbulence."

In statistical hydromechanics (^{6}), for describing the probability distributions in the corresponding mechanical systems, the method of moments, proposed in the well-known work of Friedman and Keller (^{7}), is widely used; in it, instead of the phase density ρ (in the infinite-dimensional case it does not exist), one uses a sequence of moments of various orders, i.e., the quantities $\langle u^i \rangle$, $B^{ij} =$

$\langle u^i u^j \rangle$, $B^{ijk} = \langle u^i u^j u^k \rangle$, etc., which are symmetric tensors. For these statistical characteristics, from the equations of motion (4) one obtains an infinite chain of equations, called the Friedman-Keller equations. Let us write the first equations of this chain under the assumption that $\langle u^i \rangle = 0$,

$$\Gamma^i_{\lambda\mu} B^{\lambda\mu} = 0, \tag{25}$$

$$\dot{B}^{ij} = \Gamma^i_{\lambda\mu} B^{i\lambda\mu} + \Gamma^j_{\lambda\mu} B^{i\lambda\mu}, \tag{26}$$

$$\dot{B}^{ijk} = \Gamma^i_{\lambda\mu} B^{jk\lambda\mu} + \Gamma^j_{\lambda\mu} B^{ki\lambda\mu} + \Gamma^k_{\lambda\mu} B^{ij\lambda\mu}. \tag{27}$$

For an approximate solution of this system, M. D. Millionshchikov (^{8}) proposed using a closure of the written equations by means of an additional hypothesis expressed by the relation

$$B^{ijkl} = B^{ij} B^{kl} + B^{ik} B^{jl} + B^{il} B^{jk}, \tag{28}$$

which is exactly satisfied for the Gaussian distribution.

Investigation of the Friedman-Keller chain constructed for s.g.t. shows that any stationary solution of equations (25) and (27), obtained with the aid of Millionshchikov’ s hypothesis (28), determines an exact solution of the entire chain in the form of a Gaussian distribution corresponding to the appropriate second moments. In this case $B^{ijk} = 0$, and all other odd moments are also equal to zero. The quadratic form $\sigma = \sigma_{\lambda\mu} u^\lambda u^\mu$, where the matrix σ_{ij} is inverse to B^{ij} (i.e., $\sigma_{i\lambda} B^{\lambda k} = \delta^k_i$), is an exact integral of the motion. Hence it follows at once that nontrivial solutions of the Friedman-Keller chain for s.g.t. exist only in the case when, besides the energy integral, there is a second quadratic integral of the motion.

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Note: Figure translations are in progress. See original paper for figures.