ON STOCHASTIC "HEATING" OF A NONLINEAR WAVE

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Abstract

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ON STOCHASTIC "HEATING" OF A NONLINEAR WAVE

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Recently the question of the behavior of charged particles in random fields has been intensively studied (see, for example, ([1–4])) in connection with the problem of turbulent heating of plasma. The behavior of such particles in weak random fields has the character of diffusion in velocity space, the result of which is an increase, on average, of their energy. The aim of the present work is to show that pumping of energy into a nonlinear wave by external random sources is possible, in essence analogous to the stochastic acceleration of charged particles. The effect of stochastic heating of a nonlinear wave can be observed under conditions when the damping of the latter due to collisions and other factors is small, and the spectral properties of the noises do not change with time and are determined mainly by external sources. It will be shown that a certain part of the noise energy is transferred into the energy of regular collective motion, as a result of which the energy, and with it the amplitude, of the nonlinear wave increase with time analogously to the way stochastic acceleration of particles occurs. The differences between particles and a wave in the character of the stochastic pumping of energy are connected with the spectral properties of the latter. It is established that the process of pumping energy into the wave is accelerated as its amplitude grows and can be interrupted, apparently, only by its overturning. Let us consider (without limiting the generality of the qualitative results) the influence of a random external force on stationary solutions of the Korteweg-de Vries equation, which has in a certain sense a universal character ([5]),

$$v_t + (1+v)v^x + v_{xxx} = \varepsilon F(x,t)^*, \tag{1}$$

where v is the velocity of the medium; x, t are coordinate and time; $\varepsilon F(x,t)$ is a random external force and $\varepsilon \ll 1$. The wave energy H for $\varepsilon = 0$ does not depend on time and is simultaneously the Hamiltonian of equation (1) ([6]):

$$H = \frac{1}{2}\int_{-\infty}^{+\infty} dx\left[v^2 + (v_x)^2 + \frac{v^3}{3}\right] = \pi\int_{-\infty}^{+\infty} dq\,(1-q^2)v_{qv_{-q}} +$$

$$+\frac{\pi}{3}\iiint_{-\infty}^{+\infty} dq_1\, dq_2\, dq_3\, \delta(q_1+q_2+q_3) v_{q_1} v_{q_2} v_{q_3}; \tag{2}$$

$$v_q(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, e^{-iqx} v(x,t); \qquad v_q = v^*_{-q}.$$

Equation (1) in the canonical variables v_{-q}, v_q/q has the form

$$\frac{\partial v_{-q}}{\partial t} = \frac{iq}{2\pi}\frac{\partial H}{\partial v_q}; \qquad \frac{dv_q}{dt} = -\frac{iq}{2\pi}\frac{\partial H}{\partial v_{-q}}. \tag{3}$$

* A parametric random external force can also be included in the consideration.

It is now not difficult, taking (2), (3) into account, to write the equation for the wave energy $H(t)$

$$\frac{dH}{dt} = 2\pi\varepsilon \int_{-\infty}^{+\infty} dq \left[(1-q^2) + \frac{1}{2v_q}\int_{-\infty}^{+\infty} dq_1\, v_{q_1} v_{q-q_1}\right] v_q F_{-q}; \tag{4}$$

$$F_q(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, e^{-iqx} F(x,t).$$

Let, in the absence of a perturbation, a stationary wave $v^{(0)}(x-ut)$ propagate in the medium with velocity u. In this case $v_q^{(0)} = a(q)e^{-iqut}$, and in the first approximation in ε from (4) we have

$$\frac{dH}{dt} = 2\pi\varepsilon u \int_{-\infty}^{+\infty} dq\, v_q^{(0)} F_{-q} = \varepsilon u \int_{-\infty}^{+\infty} dx\, v^{(0)}(x-ut) F(x,t), \tag{5}$$

since

$$\left[(1-q^2) + \frac{1}{2v_q^{(0)}}\int_{-\infty}^{+\infty} dq_1\, v_{q_1}^{(0)} v_{q-q_1}^{(0)}\right] = u.$$

We shall now assume that $F(x,t)$ is a stationary random process with zero mean and spectral density $S_{q\omega}$:

$$\langle F(x,t)\rangle = 0, \qquad \langle F(x,t)F(x+\xi, t+\tau)\rangle = R(\xi,\tau);$$

$$S_{q\omega} = \frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty} d\xi\, d\tau\, e^{-i(q\xi+\omega\tau)} R(\xi,\tau). \tag{6}$$

From (5) we find the mean square of the change in energy over the time t:

$$\langle [H(t)-H(0)]^2\rangle =$$

$$= (\varepsilon u)^2 \iint_{-\infty}^{+\infty} dx_1\, dx_2 \int_0^t\int_0^t dt_1\, dt_2\, v^{(0)}(x_1 - ut_1) v^{(0)}(x_2 - ut_2) R(|x_1 - x_2|, |t_1 - t_2|). \tag{7}$$

Passing in the integrand to the Fourier representation in x and t and recalling that $v_q^{(0)}(t) = a_q e^{-iqut}$, we obtain

$$\langle [H(t)-H(0)]^2\rangle = (2\pi\varepsilon u)^2 \iint_{-\infty}^{+\infty} dq\, d\omega\, a_q a_{-q} \int_0^t\int_0^t dt_1\, dt_2\, e^{i(qu+\omega)(t_1-t_2)}. \tag{8}$$

Let τ_c be the correlation time. Then, for a time $t \gg \tau_c$, after computing the integrals over t_1 and t_2, and then over ω, we obtain

$$\langle [H(t)-H(0)]^2\rangle = (2\pi\varepsilon u)^2 \iint_{-\infty}^{+\infty} dq\, d\omega\, a_q a_{-q} S_{q\omega} \frac{4\sin^2[(\omega+qu)/2]\, t}{(\omega+qu)^2}, \tag{9}$$

and finally

$$\langle [H(t)-H(0)]^2\rangle \cong t\cdot 2(2\pi)^3(\varepsilon u)^2 \int_0^\infty dq\, |a_q|^2 S_{q,qu} = tD \qquad (t \gg \tau_c), \tag{10}$$

where D is the diffusion coefficient. In particular, for a solitary wave, when

$$S_{q\omega} = \frac{1}{(2\pi)^2}\frac{1}{[1+(\omega\tau_c)^2]}$$

we have

$$D = \pi^3(24\varepsilon u)^2 \int_0^\infty dq\, \frac{q^2}{\operatorname{sh}^2\left(\pi q/\sqrt{u-1}\right)} \frac{1}{[1+(qu\tau_c)^2]},$$

which gives

$$D \approx (24\pi\varepsilon u)^2 (u-1)^{3/2} \qquad \left(\tau_c u\sqrt{u-1} \ll 1\right),$$

$$D \approx 2\pi(24\varepsilon)^2 u(u-1)/\tau_c \qquad \left(\tau_c u\sqrt{u-1} \gg 1\right).$$

With the aid of the expressions obtained, we find the time τ_0 during which the nonlinear wave overturns (at the critical velocity u_0):

$$\tau_0 \sim (u_0 - u)^2/D.$$

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Note: Figure translations are in progress. See original paper for figures.