ON THE COMPLETENESS OF SYSTEMS OF ANALYTIC FUNCTIONS GENERATED BY SUCCESSIVE DERIVATIVES OF ANALYTIC FUNCTIONS

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Abstract

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ON THE COMPLETENESS OF SYSTEMS OF ANALYTIC FUNCTIONS GENERATED BY SUCCESSIVE DERIVATIVES OF ANALYTIC FUNCTIONS

Let D be an arbitrary finite or infinite simply connected domain of the z-plane, and let $A[D]$ be the class of all functions analytic in the domain D. In addition, let the system of functions $\{\varphi_n(z)\}$, analytic in the domain D, be complete in the domain D, i.e., for every function $f(z)$ from the class $A[D]$ there exists a sequence of linear combinations

$$\sum_{n=1}^{N_k} C_{n,k}\varphi_n(z)$$

with constant coefficients, which converges uniformly to the function $f(z)$ in any closed domain $\overline{D}_1$ situated inside the domain D.

Suppose that the function $F(z,u)$ is defined in the form of the series

$$F(z,u) = \sum_{n=0}^{\infty} \varphi_n(z)u^n \tag{1}$$

under the condition that the series (1) converges uniformly for all z and u situated respectively in the given domains D and G, where G is also a simply connected domain containing the point $u = 0$.

In the present note we report results of an investigation of the completeness of the system of functions

$$[\partial^n F(z,u)/\partial u^n]_{u=\alpha_n} = \partial^n F(z,\alpha_n)/\partial u^n \tag{2}$$

under various assumptions concerning the nature of the generating function $F(z,u)$ and the sequence of complex numbers $\{\alpha_n\}$.

1. In what follows we shall use the following completeness criterion:

A system of regular functions $\{f_k(z)\}$ is complete in the domain D if every function $\varphi_m(z) \in A[D]$ of the complete system $\{\varphi_m(z)\}$ is approximated arbitrarily well in the domain D by linear combinations of the form

$$\sum_{n=}^{N_k} C_{n,k} f_n(z),$$

i.e., the inequalities hold:

$$\left| \varphi_k(z) - \sum_{n=1}^{N_k} C_{n,k} f_n(z) \right| < \varepsilon \quad (z \in D), \tag{3}$$

where $\varepsilon > 0$ is an arbitrarily small number.

In the investigation of the completeness of the system of functions (2) we use the Abel-Goncharov interpolation formula* for the function $F(z,u)$, with fixed z $(z \in D)$, with nodes

$$\alpha_0, \alpha_1, \dots, \alpha_n, \dots \quad (\alpha_n \in G,\ n = 0, 1, 2, \dots),$$

which has the form

$$F(z,u) = \sum_{k=0}^{n} \gamma_k(u) \left[\frac{\partial^k F(z,u)}{\partial u^k} \right]_{u=\alpha_k} + R_n(z,u), \tag{4}$$

* For the necessary information on the Abel-Goncharov interpolation process, see ([1]).

where

$$\gamma_k(u) = \int_{\alpha_0}^{u} dt_1 \int_{\alpha_1}^{t_1} dt_2 \cdots \int_{\alpha_{k-1}}^{t_{k-1}} dt_k, \tag{5}$$

$$R_n(z,u) = \int_{\alpha_0}^{u} dt_1 \int_{\alpha_1}^{t_1} dt_2 \cdots \int_{\alpha_n}^{t_n} \frac{\partial^{n+1} F(z, t_{n+1})}{\partial t_{n+1}^{n+1}} dt_{n+1}. \tag{6}$$

Differentiating equality (4) m times with respect to u, and then putting $u = 0$, we find

$$\varphi_m(z) = \sum_{k=m}^{n} B_{m,k} \frac{\partial^k F(z, \alpha_k)}{\partial u^k} + R_{n,m}(z), \tag{7}$$

where

$$R_{n,m}(z) = \left[\frac{\partial^m R_n(z,u)}{\partial u^m}\right]_{u=0}.$$

Hence, and from inequality (3), it is clear that the system of functions (2) is complete in that domain G in which all the series

$$\sum_{k=m}^{\infty} B_{m,k}\,\frac{\partial^k F(z,\alpha_k)}{\partial u^k} \qquad (m = 0, 1, 2, \ldots) \tag{8}$$

converge uniformly. Consequently, the conditions under which the series (8) converge uniformly are at the same time conditions for the completeness of the system of functions (2) in the domain G. By these considerations we establish certain assertions on the completeness of the system of functions (2) under various assumptions concerning the nature of the function $F(z,u)$ and the sequence of numbers $\{\alpha_n\}$. From these assertions there follows a number of results of other authors on the completeness of systems of functions of the form $\{z^n f^{(n)}(\alpha_n z)\}$ under various assumptions concerning $f(z)$ and the sequence of numbers $\{\alpha_n\}$. One of the main results of the present paper is:

Theorem 1. Let the sequence of complex numbers $\{\alpha_n\}$ be such that $|\alpha_n| \le r$, $\lim_{n\to\infty} \alpha_n = 0$, and the series $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|$ converges. Moreover, let

$$F(z,u) = \sum_{n=0}^{\infty} \varphi_n(z) u^n$$

be an analytic function of z in a finite domain D and of u in the finite disk $|u| \le r$, where r is a finite number.

Then the system of analytic functions $\{\partial^m F(z,\alpha_m)/\partial u^m\}$ is complete in the domain D.

2. In particular, if

$$f(z) = \sum_{n=0}^{\infty} C_n z^n$$

is an analytic function in the disk $|z| < R$ and $C_n = 0$ $(n = 0, 1, 2, \ldots)$, then, choosing $\varphi_n(z) = C_n z^n$, we find

$$F(z,u) = \sum_{n=0}^{\infty} C_n z^n u^n = f(zu)$$

for $|u| \le 1$. Moreover, we have

$$\frac{\partial^k F(z,\alpha_k)}{\partial u^k} = z^k f^{(k)}(\alpha_k z) \qquad (k = 0, 1, 2, ...).$$

Thus, from Theorem 1 there follows the assertion:

Corollary 1. Let $f(z)$ be an analytic function in the disk $|z| < R$, with Taylor coefficients distinct from zero, $f^{(n)}(0) \neq 0$ $(n = 0, 1, 2, ...)$, and let a sequence of complex numbers $\{a_n\}$ be such that

$$|a_n| \leq 1, \quad \lim_{n\to\infty} a_n = 0$$

and the series

$$\sum_{n=1}^{\infty} |a_n - a_{n-1}|$$

converges.

Then the system of functions $\{z^n f^{(n)}(a_n z)\}$ is complete inside the disk $|z| < R$.

This assertion was first proved by another method by A. I. Markushevich ([2]), from whose result, in the case $a_n = 1$ $(n = 0, 1, 2, ...)$, there follows the following assertion, proved by the author in ([3]).

Corollary 2. If $f(z)$ is an analytic function in the finite disk $|z| < R$, with Taylor coefficients distinct from zero, $f^{(n)}(0) \neq 0$ $(n = 0, 1, 2, ...)$, then the system of functions $\{z^n f^{(n)}(z)\}$ is complete in the same disk $|z| < R$.

3. We now consider the case where $F(z, u)$ is an analytic function of z in a finite or infinite simply connected domain D and an entire analytic function of u, i.e. the series (1) converges uniformly for all z in the domain D and for all u. In addition, the sequence of complex numbers $\{a_n\}$ has the property

$$0 < |a_0| < |a_1| < \cdots < |a_m| < \cdots < |a_n| < \cdots$$

and

$$\lim_{n\to\infty} |a_n| = \infty.$$

Theorem 2. Let $F(z, u)$ be an analytic function of z in a finite or infinite simply connected domain D and an entire function of u, with maximum modulus

$$M(F; D; r) = \max_{z\in D,\ |u|\leq r} |F(z, u)|.$$

Moreover, let $n(r)$ be the number of points of the sequence $\{S_n\}$ lying in the interval $(0, r)$, where

$$S_n = |a_0| + \sum_{k=1}^{n} |a_k - a_{k-1}|.$$

Then the system of analytic functions $\{\partial^n F(z, a_n)/\partial u^n\}$ is complete in any closed domain $\overline{D}_1$ situated inside the domain D, if the inequality

$$\log M(F; D; r/\theta) < C(\theta) n(r),$$

is satisfied, where

$$C(\theta) < \log \frac{1-\theta}{\theta}, \qquad 0 < \theta < \frac{1}{2}.$$

From Theorem 2 there follow the following assertions:

Corollary 1. Let the function $F(z, u)$ be defined by the series (1), converging uniformly for all $z \in D$ and all u, and let the functions $\varphi_n(z)$ satisfy the conditions:

$$\lim_{n\to\infty} n^{1/\rho} |\varphi_n(z)|^{1/n} = (\varepsilon\rho)^{1/\rho} |z| \qquad (z \in D).$$

Further, let $n(r)$ be the number of points of the sequence $\{S_n\}$ lying on the segment $(0, r)$, where

$$S_n = |a_0| + \sum_{k=1}^{n} |a_k - a_{k-1}|.$$

Moreover, let the numbers S_n satisfy the condition

$$\lim_{n\to\infty} \frac{n(r)}{S_n^\mu} = \nu > 0.$$

Then the system of functions $\{\partial^n F(z, \alpha_n)/\partial u^n\}$ is complete in the whole domain D for $\rho < \mu$, and is complete in the domain G_ω, which is the intersection of the domain D with the disk

$$|z| < \frac{\omega}{\omega+1} \left(\frac{\gamma}{\sigma} \log \frac{1}{\omega} \right)^{1/\rho},$$

for $\rho = \mu$, where ω is the positive root of the equation

$$\omega^\rho e^{\omega+1} = 1.$$

Corollary 2. Let

$$f(z) = \sum_{n=0}^{\infty} C_n z^n$$

be an entire analytic function with nonzero Taylor coefficients: $C_n \neq 0$ $(n = 0, 1, 2, \ldots)$, and with maximum modulus

$$M(f;r) = \max_{|z|\le r} |f(z)|.$$

Moreover, let $n(r)$ be the number of points of the sequence $\{S_n\}$ lying in the interval $(0, r)$, where

$$S_n = |\alpha_0| + \sum_{k=1}^{n} |\alpha_k - \alpha_{k-1}|.$$

Then the system of functions $\{z^n f^{(n)}(\alpha_n z)\}$ is complete in any finite disk $|z| \le R$, if the inequality

$$\log M(f; rR/\theta) < C(\theta) n(r),$$

is satisfied, where

$$C(\theta) < \log \frac{1-\theta}{\theta}, \qquad 0 < \theta < \frac{1}{2}.$$

The last assertion was proved in the author' s paper (4), written jointly with Ch. G. Atamaliyeva, from which there follows a number of concrete assertions previously proved by other authors (see also (3), when $f(z) = e^z$).

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