ON AN EXTREMAL PROPERTY OF SOLUTIONS OF ONE CLASS OF HYPERBOLIC EQUATIONS

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.946.4

MATHEMATICS

M. E. LERNER

ON AN EXTREMAL PROPERTY OF SOLUTIONS OF ONE CLASS OF HYPERBOLIC EQUATIONS

(Presented by Academician I. G. Petrovskii on 27 VI 1968)

§ 1. Consider the equation

$$\mathcal{L}[u] \equiv u_{\xi\eta} + a(\xi;\eta)u_\xi + b(\xi;\eta)u_\eta + c(\xi;\eta)u = 0 \qquad (\mathcal{L})$$

in the characteristic triangle $O_0C_0A_0$ (the domain Δ), adjacent from below to the descending Jordan arc O_0A_0 (the "coordinate" η is a strictly decreasing function of the "abscissa" ξ, $\eta_{O_0} > \eta_{A_0}$). We shall assume that the coefficients of equation $(\mathcal{L})$ are continuous together with $a_\xi(\xi;\eta)$ in $\overline{\Delta} \setminus O_0A_0$ and satisfy there certain conditions

$$(K). \quad 1)\ a > 0; \quad 2)\ h \equiv a_\xi + ab - c > 0; \quad 3)\ c \geq 0.$$

Theorem 1 (maximum principle of absolute-extremum type).
Let: 1) the function $u(\xi;\eta)$ be continuous in $\overline{\Delta}$; 2) $\mathcal{L}[u] \equiv 0$ in Δ; 3) $u \in C^{(2)}(\Delta)$; 4) $u \in C^{(1)}(\overline{\Delta} \setminus \overline{O_0A_0})$; 5) $u|_{O_0C_0} \equiv 0$.

Then $\max_{\overline{\Delta}} u$, if it is positive, is attained on the characteristic $\overline{C_0A_0}$.

Proof. Suppose the contrary. Let $\max_{\overline{\Delta}} u > 0$, but it is not attained on $\overline{C_0A_0}$. Then, by the lemma of work (1), p. 256, $\max u$ cannot be attained in Δ and, consequently, is attained at some interior point Q of the arc O_0A_0. In view of the continuity of $u(\xi;\eta)$ in $\overline{\Delta}$, in some neighborhood of the point Q there exists a point Q' at which

$$u(Q') > \max_{C_0A_0} u. \qquad (*)$$

In the domain Δ, through the point Q', draw a descending arc supported on the characteristics O_0C_0 and C_0A_0, respectively, at the points O_0' and A_0'. Denote by Δ' the open triangular domain $C_0A_0'O_0'$. By Theorem 1 of ([1]), $\max_{\overline{\Delta}} u$ is attained only on $\overline{C_0A_0'}$. Consequently,

$$u(Q') < \max_{C_0A_0'} u \leq \max_{C_0A_0} u,$$

which contradicts the inequality $(*)$ and proves the theorem.

It is easy to show that in Theorem 1 the requirements of continuity on C_0A_0 of the coefficients of equation $(\mathcal{L})$, u_ξ, u_η are superfluous; moreover, the requirement $\mathcal{L}[u] \equiv 0$ in Δ can be replaced by the condition $\mathcal{L}[u] \leq 0$.

Corollary 1 (modified maximum principle of absolute-extremum type). *Under the conditions of Theorem 1,* $\max_{\overline{\Delta}} |u|$ *is attained on* $\overline{C_0A_0}$.

§ 2. Consider the equation

$$K_m[u] \equiv u_{xx} + \operatorname{sgn} y \cdot |y|^m u_{yy} + M(x;y)u_x + N(x;y)u_y + F(x;y)u = 0 \quad (K_m)$$

in the domain D, bounded by: 1) a simple Jordan arc σ, lying in the upper half-plane and resting on the axis $y = 0$ at the points $O(0;0)$ and $A(a;0)$, $a > 0$; 2) the characteristics OC and AC in the lower half-plane. Let $D_1 \equiv D \cap (y > 0)$, $D_2 \equiv D \cap (y < 0)$. The coefficients of the equation (K_m) are as follows: $M, N \in C^{(1)}(D_1 \cup D_2 \cup \overline{OC})$, $F \in C^{(0)}(D_1 \cup D_2 \cup OC)$, $F \leq 0$ in D_1.

Theorem 2. *Let the coefficients of the equation* (K_m) *be such that the coefficients of the corresponding equation of the form* $(\mathcal{L})$ *satisfy the conditions*

$$(K) \left(\xi = x - \frac{2}{2-m}(-y)^{(2-m)/2},\ \eta = -x - \frac{2}{2-m}(-y)^{(2-m)/2} \right)$$

and suppose that: 1) the function $u(x;y)$ *is continuous in* $\overline{D}$*; 2)* $K_m[u] \equiv 0$ *in* $D_1 \cup D_2$*; 3)* $u \in C^{(2)}(D_1 \cup D_2)$*; 4)* $u \in C^{(1)}[\overline{D} \setminus (\overline{OA} \cup \overline{AC})]$*; 5)* $u|_{OC} \equiv 0$.

Then: 1) $\max_{\overline{D_2}} u$*, if it is positive, cannot be attained in* D_2 *and is attained on the characteristic* $\overline{AC}$*; 2)* $\max_{\overline{D}} u$ *cannot be attained in* $D_1 \cup D_2$ *and is attained on* $\overline{\sigma} \cup AC$.

Proof. I. Assertion 1) follows directly from Theorem 1.

II. Let $\max_{\overline{D}} u > 0$. By a known property of elliptic equations and by the first assertion of the theorem, $\max_{\overline{D}} u$ cannot be attained, respectively, in D_1 and D_2. If it is attained on $\overline{OA}$, then by the first assertion of the theorem it is attained on $\overline{AC}$. Thus the theorem is proved.

Theorem 3. *Under the conditions of Theorem 2, for the equation* (K_m)*, the solution of each of the problems of the type of Frankl' s "shock" problem, posed in work (2) for the general Lavrent' ev–Bitsadze equation, is unique.*

Proceeding from work (1), it is easy to show that there is a broad class of equations of the form (K_m) for which the assertions of Theorems 2 and 3 are valid.

§ 3. Consider the equations (K_{nn}) and (G_{mn}) with two mutually perpendicular lines of parabolic degeneration, respectively of the second and of the first and second kind,

$$K_{nn}[u] \equiv \operatorname{sgn} x \cdot |x|^n u_{xx} + \operatorname{sgn} y \cdot |y|^n u_{yy} = 0, \qquad 0 < n < 2, \qquad (K_{nn})$$

$$G_{mn}[u] \equiv \operatorname{sgn} x \cdot \operatorname{sgn} y \cdot |x|^n |y|^m u_{xx} + u_{yy} = 0, \quad m = \frac{n}{1-n}, \quad 0 < n < 1 \; (G_{mn})$$

in the domain D corresponding to each of them, bounded by: 1) a simple Jordan arc σ, lying in the first quadrant and resting on the coordinate axes at the points $A(a, 0)$ and $B(0; b)$; 2) two pairs of characteristics OC and AC, OE and BE.

Theorem 4. *Suppose: 1) the function $u(x; y)$ is continuous in $\overline{D}$; 2) $K_{nn}[u] \equiv 0$ in $D \setminus (OA \cup OB)$; 3) $u \in C^{(2)}[D \setminus (OA \cup OB)]$; 4) $u \in C^{(1)}[\overline{D} \setminus (OA \cup AC \cup OB \cup BE)]$; 5) $u|_{OC \cup OE} \equiv 0$.*

Then $\max_{\overline{D}} u$, if it is positive ($\min_{\overline{D}} u$, if it is negative), is attained on $\overline{\sigma} \cup AC \cup BE$ and cannot be attained in $D \setminus (OA \cup OB)$.

The proof is analogous to the proof of Theorem 2 and follows from Theorem 1 and a known property of elliptic equations.

Theorem 5. *Suppose: 1) the function $u(x; y)$ is continuous in $\overline{D}$; 2) $G_{mn}[u] \equiv 0$ in $D \setminus (OA \cup OB)$; 3) $u \in C^{(2)}[D \setminus (OA \cup OB)]$; 4) $u \in C^{(1)}[\overline{D} \setminus (\overline{OB} \cup \overline{BE} \cup A)]$; 5) $u|_{OC \cup OE} \equiv 0$.*

Then $\max_{\bar{D}} u$, if it is positive ($\min_{\bar{D}} u$, if it is negative), is attained on $\sigma \cup BE$ and cannot be attained in $D \setminus OB$.

The proof follows easily from Theorem 1, the maximum principle for elliptic equations, and Theorem 1 ([3]).

Problem 1. Suppose a one-to-one correspondence is established between all points of the characteristics AC and BE ($AC \cup BE$) and a certain set $E \subset \sigma$, and let Q and Q' be arbitrary corresponding points ($Q \sim Q'$, $Q \in E$, $Q' \in \overline{(AC \cup BE)}$; $E' \equiv \sigma \setminus E$).

Find a function $u(x; y)$ satisfying conditions 1)–4) of Theorem 4 and the boundary conditions: 1) $u|_{OC \cup OE} = \varphi_1$; 2) $u|_{E'} = \varphi_2$; 3) $\partial u / \partial n|_E = \varphi_3$; 4) $u(Q) - u(Q') = g(Q')$.

Here, as below, $\varphi_1, \varphi_2, \varphi_3, g$ are prescribed continuous functions; n is the interior normal.

Problem 2. Suppose a one-to-one correspondence is established between all points of the characteristic BE and a certain set $E \subset \sigma$, $Q \sim Q'$, $Q \in E$, $Q' \in BE$, $\sigma' \equiv \sigma \setminus E$.

Find a function $u(x;y)$ satisfying conditions 1)–4) of Theorem 5 and the boundary conditions: 1) $u|_{OC \cup OE} = \varphi_1$; 2) $u|_{E'} = \varphi_2$; 3) $\partial u/\partial n|_E = \varphi_3$; 4) $u(Q) - u(Q') = g(Q')$.

From Theorems 4 and 5, respectively, Theorems 6 and 7 follow easily.

Theorem 6. For equation (K_{nn}), the solution of Problem 1 is unique.

Theorem 7. For equation (G_{mn}), the solution of Problem 2 is unique.

We note that in Problems 1 and 2 the boundary condition 4) determines a "curved jump of compactification" on an disconnected set.

Kuibyshev Polytechnic Institute
named after V. V. Kuibyshev

Received
18 VI 1968

CITED LITERATURE

([1]) M. E. Lerner, *Volga Mathematical Collection*, vol. 3, 255, Kuibyshev, 1965. ([2]) M. E. Lerner, *Volga Mathematical Collection*, vol. 5, 185, Kazan, 1966. ([3]) M. E. Lerner, DAN, 177, No. 6, 1269 (1967).

Note: Figure translations are in progress. See original paper for figures.