ON SYMMETRIC CONTINUITY AND SYMMETRIC DIFFERENTIABILITY WITH RESPECT TO SETS

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Abstract

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ON SYMMETRIC CONTINUITY AND SYMMETRIC DIFFERENTIABILITY WITH RESPECT TO SETS

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1. Let $f(x)$ be a real function defined on the interval $(0,1)$; let k be an arbitrary fixed natural number;

$$\Delta^k f(x,h) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f[x + (k-2i)h]$$

is the symmetric difference of order k at the point x; and let Q be a set of points h on the line having zero as a limit point and situated to the right of zero.

We shall call $f(x)$ Q-symmetrically continuous of order k at the point $x \in (0,1)$ if

$$\lim_{h\to 0,\ h\in Q} \Delta^k f(x,h) = 0.$$

We shall call $f(x)$ Q-symmetrically differentiable of order k at the point $x \in (0,1)$ if there exists

$$D_Q^k f(x) = \lim_{h\to 0,\ h\in Q} \frac{\Delta^k f(x,h)}{(2h)^k}.$$

The upper and lower Q-symmetric derivative numbers are defined as follows:

$$\overline{D}_Q^k f(x) = \overline{\lim}_{h\to 0,\ h\in Q} \frac{\Delta^k f(x,h)}{(2h)^k}, \qquad \underline{D}_Q^k f(x) = \underline{\lim}_{h\to 0,\ h\in Q} \frac{\Delta^k f(x,h)}{(2h)^k}.$$

If zero is an interior point for the set Q, then Q-symmetric continuity of order k coincides with the usual symmetric continuity of order k, while the definition of the Q-symmetric derivative coincides with the definition of the Riemann derivative of order k. Recall that a function $f(x)$ has a Riemann derivative of order k at the point x if there exists

$$D^k f(x) = \lim_{h\to 0} \frac{\Delta^k f(x,h)}{(2h)^k}.$$

For $k = 1$ this definition coincides with the definition of the symmetric derivative; for $k = 2$, with the definition of the Schwarz derivative.

It is clear that ordinary continuity at a point x implies Q-symmetric continuity of any order k, and ordinary differentiability implies Q-symmetric differentiability of order 1 at this point x, whatever the set Q may be. The converse (for a fixed set Q) does not hold. However, as follows from the theorems given below, if one assumes that Q-symmetric continuity or Q-symmetric differentiability of order k of the function $f(x)$ holds at every point of a set E of positive measure, then, under certain conditions imposed on Q, it follows that $f(x)$ is ordinarily continuous or ordinarily differentiable almost everywhere on E.

We shall assume that for all x the set Q is one and the same. The idea of such a generalization is due to G. Kh. Sindalovskii. He introduced (^{1}) the concepts of continuity and differentiability with respect to congruent sets (the concepts of Q-continuity and Q-differentiability) and studied the connection of these concepts with the concepts of ordinary continuity and differentiability.

Let us introduce the following classes of sets Q having zero as their limit point.

To class (A) we assign those sets Q that have positive measure in every neighborhood of zero, i.e. such that $\operatorname{mes} Q \cap (0,\delta) > 0$ for every $\delta > 0$.

To class (B) we assign those sets Q that have positive lower density at zero, i.e. those for which

$$\varliminf_{\delta\to 0} \frac{\operatorname{mes} Q \cap (0,\delta)}{\delta} > 0.$$

To class (C) we assign those sets Q whose closure $\overline{Q}$ belongs to class (B).

Theorem 1. Let k be an arbitrary fixed natural number, $f(x)$ a function measurable on $(0,1)$, for which

$$\lim_{\substack{h\to 0\\ h\in Q}} \Delta^k f(x,h) = 0$$

at every point of a set E, $E \subset (0,1)$, $\operatorname{mes} E > 0$, and let the set Q belong to class (A). Then $f(x)$ is continuous almost everywhere on E.

For every set Q not belonging to class (A), there exists an everywhere on $(0,1)$ symmetrically discontinuous* (and hence everywhere discontinuous in the ordinary sense) measurable function that will be Q-symmetrically continuous of order k almost everywhere on $(0,1)$.

Theorem 2. Let k be an arbitrary fixed natural number, and let the set Q belong to class (B). If $f(x)$ is a function measurable on $(0,1)$ for which

$$\lim_{\substack{h\to 0\\ h\in Q}} \left|\Delta^k f(x,h)/(2h)^k\right| < +\infty$$

and $f'_{ac}(x)$ exists at every point $x \in E,\ E \subset (0,1),\ \operatorname{mes} E > 0$, then $f(x)$ is differentiable in the ordinary sense almost everywhere on E.

For every set Q not belonging to class (B), there exists a function measurable on $(0,1)$ that has a Q-symmetric derivative of order k almost everywhere on $(0,1)$ and nowhere has a symmetric (and hence ordinary) derivative.

Theorem 3. Let k be an arbitrary fixed natural number, and let the set Q belong to class (C). If $f(x)$ is a function continuous on $(0,1)$ for which

$$\lim_{\substack{h\to 0\\ h\in Q}} \left|\frac{\Delta^k f(x,h)}{(2h)^k}\right| < +\infty$$

and $f'_{ac}(x)$ exists at every point $x \in E,\ E \subset (0,1),\ \operatorname{mes} E > 0$, then $f(x)$ is differentiable in the ordinary sense almost everywhere on E.

For every set Q not belonging to class (C), one can construct a function continuous on $(0,1)$ having a finite Q-symmetric derivative of order k on some set $E,\ E \subset (0,1)$, and having no ordinary derivative on it. The measure of the set E may be arbitrarily close to one.

For $k = 1$ the following assertions are valid:

Theorem 4. Let $f(x)$ be a function measurable on $(0,1)$; Q a set belonging to class (B). Then $\overline{D}_Q f(x)$ almost everywhere on $(0,1)$ is equal to $\overline{D} f(x)$.

For every set Q not belonging to class (B), there exists a function $\varphi(x)$, measurable on $(0,1)$ and nowhere symmetrically differentiable, for which almost everywhere on $(0,1)$ the relation

$$\overline{D}_Q \varphi(x) < \overline{D}\varphi(x) = +\infty$$

holds (here $\overline{D} f(x)$ is the upper symmetric derivative number of the function $f(x)$).

Theorem 5. Let $f(x)$ be a function continuous on $(0,1)$; Q a set— a set belonging to class (C). Then almost everywhere on $(0,1)$

$$\overline{D}_Q f(x) = \overline{D} f(x).$$

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* A function $f(x)$ is called everywhere symmetrically discontinuous on $(0,1)$ if for every $x \in (0,1)$ $\Delta^1 f(x,h)$ does not tend to zero as $h \to 0$.

For every set Q not belonging to class (C), there exists a continuous function $\varphi(x)$, nowhere differentiable on $(0,1)$, for which almost everywhere the relation

$$\overline{D}_Q\varphi(x) < \overline{D}\varphi(x) = +\infty$$

holds.

Analogous assertions hold for the lower symmetric derived numbers.

Theorem 6. Let $f(x)$ be a measurable function on $(0,1)$; let Q be a set belonging to class (B). If at each point $x \in E \subset (0,1)$ at least one of the inequalities

$$\overline{D}_Q f(x) < +\infty$$

or

$$\underline{D}_Q f(x) > -\infty$$

holds, then $f(x)$ is differentiable in the ordinary sense almost everywhere on E.

For every set Q not belonging to class (B), there exists a measurable symmetric function $\varphi(x)$, nowhere symmetrically differentiable on $(0,1)$, for which almost everywhere on $(0,1)$ the inequality

$$\overline{D}_Q\varphi(x) < +\infty$$

holds.

Theorem 7. Let $f(x)$ be a continuous function on $(0,1)$; let Q be a set belonging to class (C). If at each point $x \in E \subset (0,1)$ at least one of the inequalities

$$\overline{D}_Q f(x) < +\infty$$

or

$$\underline{D}_Q f(x) > -\infty$$

holds, then $f(x)$ is differentiable in the ordinary sense almost everywhere on E.

For every set Q not belonging to class (C), there exists a function $\varphi(x)$ continuous on $(0,1)$, having no ordinary derivative, but satisfying the inequality

$$\overline{D}_Q\varphi(x) < +\infty$$

almost everywhere on $(0,1)$.

2. The proof of Theorem 2 is based on the following assertions:

Lemma 1. Let k be a fixed natural number; let $f(x)$ be a function measurable in the sense of Lebesgue on $(0,1)$; let Q be a set belonging to class (B); and let F be a Borel-measurable set such that: a) $F \subset (0,1)$, $\operatorname{mes} F = 1$; b) $f(x)$ is Borel-measurable on F; c) for every $x \in CF$ ($CF = (0,1) \setminus F$) there exists a sequence of numbers $\{c_i\}$, $c_i \in F$, such that $c_i \to x$ and $f(c_i) \to f(x)$ as $i \to \infty$. If

$$\overline{D}^k_Q f(x) < +\infty$$

for $x \in E$, $E \subset (0,1)$, $\operatorname{mes} E > 0$, then there exist a perfect set P, $P \subset E \cap F$, of measure arbitrarily close to the measure of E, and numbers $\eta_0 > 0$, $l_0 > 0$, such that

$$\Delta^k f(x,h)/(2h)^k < \eta_0$$

for $x \in P$, $h \in Q$, $h < l_0$, such that $x \pm rh \in F$ $(1 \leq r \leq k)$.

Lemma 1 is proved by the methods developed by G. Kh. Sindalovskii (^{1,2}). The first part of Theorem 3 follows from the first part of Theorem 2. Theorem 4 is based on the following assertion:

Lemma 2. Let $f(x)$ be a measurable function on $(0,1)$; let Q be a set belonging to class (B). If

$$\overline{D}_Q f(x) < +\infty$$

on some set E of positive measure, then almost everywhere on E $f(x)$ is asymptotically differentiable. An analogous assertion holds if

$$\underline{D}_Q f(x) > -\infty.$$

Theorems 5 and 6 follow from Theorem 4; Theorem 7 follows from Theorem 6.

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CITED LITERATURE

1. G. Kh. Sindalovskii, *Izv. AN SSSR, Ser. Matem.*, **26**, 125 (1962).
2. G. Kh. Sindalovskii, *Izv. AN SSSR, Ser. Matem.*, **24**, 707 (1960).

Note: Figure translations are in progress. See original paper for figures.

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