MATHEMATICS

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Abstract

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ON THE SPECTRUM OF SINGULAR INTEGRAL OPERATORS IN WEIGHTED L_p SPACES

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1. Let Γ be a simple smooth closed oriented curve in the plane, enclosing the point $\lambda = 0$. Let $t_1, \dots, t_n$ be certain points on Γ, and let $p, \beta_1, \dots, \beta_n$ be real numbers satisfying the relations

$$1 < p < \infty; \qquad -1 < \beta_k < p - 1 \quad (k = 1, 2, \dots, n). \tag{1}$$

By $L_p(\Gamma, \rho)$ we denote the space L_p on Γ with weight

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}.$$

The following important proposition is due to B. V. Khvedelidze (^{1}), p. 24: if the numbers $p, \beta_1, \dots, \beta_n$ satisfy conditions (1), then the singular integration operator S, defined by the equality

$$(S\varphi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} \, d\tau \quad (t \in \Gamma),$$

is a linear bounded operator in the space $L_p(\Gamma, \rho)$.

In the present note singular integral operators of the following two forms are considered in $L_p(\Gamma, \rho)$:

$$(A\varphi)(t) = c(t)\varphi(t) + d(t)(S\varphi)(t); \qquad (B\varphi)(t) = c(t)\varphi(t) + (Sd\varphi)(t),$$

where $c(t), d(t)$ $(t \in \Gamma)$ are piecewise-continuous functions.

If the coefficients $c(t)$ and $d(t)$ are continuous functions, then, as is known, the spectra of the operators A and B coincide and do not depend on the choice of the numbers $p, \beta_1, \dots, \beta_n$ satisfying condition (1). In the case under consideration this spectrum consists of all points of the curves $\lambda = c(t) + d(t)$ and $\lambda = c(t) - d(t)$, and also of the points λ not lying on these curves for which the number

$$\varkappa(\lambda) = \operatorname{ind}\left[(c(t) + d(t) - \lambda)/(c(t) - d(t) - \lambda)\right] \neq 0.$$

At the points λ satisfying the last condition, the operators $A - \lambda I$, $B - \lambda I$ are invertible on the left, on the right, or on both sides, depending on whether the number $\varkappa(\lambda)$ is positive, negative, or equal to zero.

This proposition ceases to be true if the coefficients $c(t)$ and $d(t)$ are not continuous.

In the present note the spectrum of the operators A and B in $L_p(\Gamma, \rho)$ is found in the case when the coefficients $c(t)$ and $d(t)$ are piecewise-continuous functions. The case of open contours is also considered. The results obtained generalize the results of the authors (^{2}, ^{3}), established for the case of closed contours and weight $\rho = 1$.

2. Let z and ζ be two points of the complex plane and let p $(2 < p < \infty)$ be a certain real number. By $v_p(z, \zeta)$ we denote the arc of a circle joining the points z, ζ and possessing the following two properties: a) from the interior points of the arc $v_p(z, \zeta)$, the segment joining the points z and ζ is seen at an angle $2\pi/p$; b) the direction from z to ζ along the arc $v_p(z, \zeta)$ is counterclockwise. In the case $1 < p < 2$ we put $v_p(z, \zeta) =$

$= v_q(\zeta, z)$ $(p^{-1} + q^{-1} = 1)$, and in the case $p = 2$, by $v_2(z, \zeta)$ we denote the segment joining the points z and ζ. Denote by Λ the set of all piecewise-continuous functions, continuous from the left, on the contour Γ.

Let $a(t) \in \Lambda$, and let $t_1, \dots, t_n$ be all its points of discontinuity. Let $\omega = (p, \beta_1, \dots, \beta_n)$ denote an arbitrary vector with coordinates satisfying relations (1). To the function $a(t)$ and the vector ω we associate a continuous closed naturally oriented curve $V_\omega(a)$, obtained by adding to the set of values of the function $a(t)$ n arcs $v_{\delta_k}(a(t_k), a(t_k + 0))$, where $\delta_k = p/(1 + \beta_k)$.*

We shall call the function $a(t)$ $(\in \Lambda)$ ω-nonsingular if $0 \notin \overline{V_\omega(a)}$ (cf. (^{3})).

The **index** (more precisely, the ω-index) of an ω-nonsingular function is defined to be the number of turns of the curve $V_\omega(a)$ about the point $\lambda = 0$, i.e.

$$\operatorname{ind}_\omega(a) = \frac{1}{2\pi} \oint_{V_\omega(a)} d(\arg t).$$

Theorem 1. Let $c(t)$ and $d(t) \in \Lambda$; let $t_1, \dots, t_n$ be all the points of discontinuity of these functions; let the vector $\omega = (p, \beta_1, \dots, \beta_n)$ satisfy conditions (1), and let

$$\rho(t) = \prod |t - t_k|^{\beta_k}.$$

In order that the operator

$$A = c(t)I + d(t)S \qquad (A = c(t)I + Sd(t)I) \tag{2}$$

be a Φ_+- or Φ_--operator in the space $L_p(\Gamma, \rho)$, it is necessary and sufficient that the following two conditions hold:

I. $\inf |b(t)| > 0$ $(t \in \Gamma)$.

II. The function $a(t)/b(t)$ is ω-nonsingular, where $a(t) = c(t) + d(t)$ and $b(t) = c(t) - d(t)$.

If conditions I and II are satisfied and $\chi = \operatorname{ind}_\omega[a(t)/b(t)]$, then:

1) for $\chi > 0$ the operator A is left-invertible in $L_p(\Gamma, \rho)$ and $\dim \operatorname{coker} A = \chi$;

2) for $\chi < 0$ the operator A is right-invertible in $L_p(\Gamma, \rho)$ and $\dim \ker A = -\chi$;

3) for $\chi = 0$ the operator A is invertible in $L_p(\Gamma, \rho)$.

For the case of the space L_p without weight (i.e. $\rho = 1$) this theorem was established in (^{3}).

It is not difficult to verify that conditions I and II are equivalent to the following:

I′. $\inf |a(t)| > 0$ and $\inf |b(t)| > 0$ $(t \in \Gamma)$;

II′. $\beta_k \neq p\alpha_k - 1$ $(k = 1, 2, \dots, n)$, where

$$\alpha_k = (1/2\pi) \arg(a(t_k) b(t_k + 0)/[b(t_k) a(t_k + 0)]), \quad 0 \leq \alpha_k < 1.$$

In this formulation of the conditions of Theorem 1 it becomes clear that, in its sufficient part, Theorem 1 generalizes some results from the monograph of B. V. Khvedelidze (^{1}). In that monograph Noether theorems are proved for the operator A in the space $L_p(\Gamma, \rho)$ under the assumption I′ and the existence of a certain special relation among the numbers β_k, α_k, p, ensuring, in particular, the fulfillment of condition II′.

From Theorem 1 the following theorem on the spectrum of the operator A, defined by equality (2), is easily derived.

Theorem 2. Suppose that the first conditions of Theorem 1 are satisfied. Then the complement $C\Phi_A$ to the Φ-set** Φ_A of the operator A in the space $L_p(\Gamma, \rho)$ consists of the union of the sets of values of the functions $a(t)$ and $b(t)$ and the set of complex numbers λ satisfying, for at least one pair of numbers k and μ $(k = 1, \dots, n;\ 0 \leq \mu \leq 1)$, the equation

$$(a(t_k) - \lambda)(b(t_k + 0) - \lambda) E_k(\mu) + (a(t_k + 0) - \lambda)(b(t_k) - \lambda) F_k(\mu) = 0,$$

* The orientation of the curve $V_\omega(a)$ is chosen so that on intervals of continuity of the function $a(t)$ the motion along the curve $V_\omega(a)$ is determined by the motion of the variable t on Γ in the positive direction, and along the arcs $v_{\delta_k}(a(t_k), a(t_k + 0))$ from the point $a(t_k)$ to the point $a(t_k + 0)$.

** The Φ-set of an operator A is the set Φ_A of all Φ-points of this operator (see (^{4})).

where

$$E_k(\mu) = e^{i\theta} - e^{i\mu\theta}, \quad F_k(\mu) = e^{i\mu\theta} - 1 \; (\theta = 2\pi - 4\pi(1+\beta_k)/p)$$

for $p > 2(1+\beta_k)$,

$$E_k(\mu) = e^{i\mu\theta} - 1, \quad F_k(\mu) = e^{i\theta} - e^{i\mu\theta} \; (\theta = -2\pi + 4\pi(1+\beta_k)/p)$$

for $p < 2(1+\beta_k)$,

$$E_k(\mu) = \mu, \quad F_k(\mu) = 1 - \mu \quad \text{for } p = 2(1+\beta_k).$$

The spectrum of the operator A in $L_p(\Gamma, \rho)$ consists of all points of $C\Phi_A$ and of points $\lambda \in \Phi_A$ for which

$$\operatorname{ind}_\omega[(a-\lambda)/(b-\lambda)] \neq 0.$$

All the results presented admit a natural generalization to the case where Γ consists of a finite number of closed smooth contours (see (3)).

3. Let now Γ be an oriented curve consisting of a finite number of smooth simple open arcs. Complete the contour Γ, preserving its orientation, to a closed curve $\tilde{\Gamma}$ satisfying the conditions of the preceding paragraph. Denote by $\widetilde{S}$ (S) the operator of singular integration along the contour $\tilde{\Gamma}$ (Γ). To each operator $A = c(t)I + d(t)S$ $(A = cI + SdI)$, where $c(t)$ and $d(t)$ are measurable bounded functions on Γ, acting in the space $L_p(\Gamma, \rho)$ $(1 < p < \infty)$, we associate the operator $\widetilde{A} = \tilde{c}(t)I + \tilde{d}(t)\widetilde{S}$ $(\widetilde{A} = \tilde{c}I + \widetilde{S}\tilde{d}I)$, acting in the space $L_p(\tilde{\Gamma}, \rho)$; here $\tilde{c}(t)$ and $\tilde{d}(t)$ denote the functions defined by the equalities

$$\tilde{c}(t) = \begin{cases} c(t), & t \in \Gamma, \\ 1, & t \in \tilde{\Gamma} \setminus \Gamma; \end{cases} \qquad \tilde{d}(t) = \begin{cases} d(t), & t \in \Gamma, \\ 0, & t \in \tilde{\Gamma} \setminus \Gamma. \end{cases}$$

Embed the space $L_p(\Gamma, \rho)$ in the space $L_p(\tilde{\Gamma}, \rho)$, by taking all functions from $L_p(\Gamma, \rho)$ to be equal to zero on the additional contour $\tilde{\Gamma} \setminus \Gamma$. Denote by $\mathcal{L}$ the subspace of $L_p(\tilde{\Gamma}, \rho)$ consisting of all functions vanishing on the contour Γ. Represent each function $f(t) \in L_p(\tilde{\Gamma}, \rho)$ as a vector $f(t) = \{f_1(t), f_2(t)\}$, where $f_1(t) \in L_p(\Gamma, \rho)$ and $f_2(t) \in \mathcal{L}$. Then the operator $\widetilde{A}$ can be represented in the form of a second-order operator matrix

$$\widetilde{A} = \begin{pmatrix} A & A_{12} \\ 0 & I_{\mathcal{L}} \end{pmatrix}, \tag{3}$$

where $I_{\mathcal{L}}$ is the identity operator in the subspace $\mathcal{L}$. Hence it is easily derived that

Lemma. The subspace $L_p(\Gamma, \rho)$ is invariant with respect to the operator $\widetilde{A}$. The operator A is invertible from one side if and only if the operator $\widetilde{A}$ is invertible from the same side. If $\widetilde{A}^{-1}$ is the operator inverse to $\widetilde{A}$ from one side, then the restriction of the operator $\widetilde{A}^{-1}$ to $L_p(\Gamma, \rho)$ is inverse to A from the same side.

In order that the operator A be a Φ-(Φ_+, Φ_-) operator, it is necessary and sufficient that the operator $\widetilde{A}$ be such, and moreover

$$\dim \ker A = \dim \ker \widetilde{A}, \qquad \dim \operatorname{coker} A = \dim \operatorname{coker} \widetilde{A}.$$

This lemma makes it possible to reduce completely the study of singular integral equations in the case of open contours to the case of closed contours. In particular, if the coefficients $c(t)$ and $d(t)$ are piecewise continuous, then the lemma just formulated makes it easy to carry over Theorems 1, 2 to the case of open contours, and also to the mixed case, when the contour Γ consists of both closed and open lines. For lack of space we shall not dwell on the formulations of the corresponding propositions.*

*

We note that the lemma presented makes it possible without difficulty to extend to the case of open contours and the mixed case the results of I. B. Simonenko (5) (see also (2)) on singular integral equations with measurable bounded coefficients.

We note that the method given above for reducing the case of open contours to that of closed ones under other restrictions was considered by F. D. Gakhov (see (6)).

4. In this section the results of the preceding section will be illustrated for the case $A = S$ (i.e., $c(t) \equiv 0,\ d(t) \equiv 1$).

Let Γ be a contour consisting of m simple smooth oriented open arcs. Denote by $c_1, \dots, c_{2m}$ the endpoints of these arcs, and by $\rho(t)$ the weight defined by the equality $\rho(t) = \prod |t - c_k|^{\beta_k}$, where $1 < p < \infty$, $-1 < \beta_k < p - 1$.

Theorem 3. In order that the operator S be a Φ_+- or Φ_--operator in the space $L_p(\Gamma, \rho)$, it is necessary and sufficient that $1 + \beta_k \neq p/2$ for all $k = 1, \dots, 2m$.

Let $1 + \beta_k \neq p/2$ for all k, and let n be the number of points c_k at which $1 + \beta_k > p/2$; then:

1) if $m > n$, the operator S is right-invertible in $L_p(\Gamma, \rho)$ and $\dim \ker S = m - n$;

2) if $m < n$, the operator S is left-invertible in $L_p(\Gamma, \rho)$ and $\dim \operatorname{coker} S = n - m$;

3) if $m = n$, the operator S is invertible in $L_p(\Gamma, \rho)$.

The complement $C\Phi_S$ of the Φ-set of the operator S is the union of $2m$ arcs Z_k of circles, each of which joins the points -1 and 1 and passes through one of the corresponding points $i \operatorname{ctg}[\pi(1+\beta_k)/p]$ or $-i \operatorname{ctg}[\pi(1+\beta_k)/p]$, depending on whether c_k is the end or the beginning of the corresponding arc.

Assertions 1)–3) of Theorem 3 for the case when the numbers β_k take one of the two values $2p$ or $2p/(1+p)$ were proved by B. V. Khvedelidze ((1) p. 43).

For the case when the line Γ is located on the real axis and $\beta_k = 0$, it was shown in (7) that the arcs Z_k belong to the spectrum of the operator S.

In conclusion, we note that all the results of the note carry over to more general symmetric spaces with weights (cf. (3)), for example to certain Orlicz spaces with weights.

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Note: Figure translations are in progress. See original paper for figures.

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