EXPONENTIAL DECAY OF THE SOLUTION OF THE WAVE EQUATION IN THE EXTERIOR OF AN ALMOST STAR-SHAPED DOMAIN

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 517.946

MATHEMATICS

V. Ya. Ivrii

EXPONENTIAL DECAY OF THE SOLUTION OF THE WAVE EQUATION IN THE EXTERIOR OF AN ALMOST STAR-SHAPED DOMAIN

(Presented by Academician S. L. Sobolev on 12 V 1969)

1. In this paper we study the exponential decay as $t \to \infty$ of the solution of the mixed problem

$$u_{tt} - \Delta u = 0, \qquad x \in C\overline{B}, \quad t > 0, \tag{1}$$

$$u|_{\partial B} = 0, \qquad t > 0, \tag{2}$$

$$u|_{t=0} = f_0, \qquad u_t|_{t=0} = f_1, \qquad x \in CB \quad (f_0|_{\partial B} = 0), \tag{3}$$

where B is an open bounded domain of odd-dimensional Euclidean space R^l $(l > 1)$, $x = (x_1, \dots, x_l)$ are the spatial variables; CB, the complement of $\overline{B}$, is connected; ∂B, the boundary of B, is of class C^2; B contains the origin. Let R be such that the ball $S(R) = \{x : \ |x| < R\}$ contains $\overline{B}$. Introduce the energy norms

$$\|u\|^2_{E,t} = \int_{CB} \left(|u_t|^2 + \sum_{i=1}^{l} |u_{x_i}|^2 \right) dx,$$

$$\|u\|^2_{E,t,R} = \int_{CB \cap S(R)} \left(|u_t|^2 + \sum_{i=1}^{l} |u_{x_i}|^2 \right) dx$$

and the norm on the set of initial data $f = \{f_0, f_1\}$

$$\|f\|^2_{E} = \int_{CB} \left(|f_1|^2 + \sum_{i=1}^{l} |f_{0x_i}|^2 \right) dx.$$

Define the spaces of initial data H and $H(R)$ as the closures in the energy norm $\|\cdot\|_E$ of the sets $C_0^\infty(CB)$ and $C_0^\infty(CB \cap S(R))$, respectively.

It is known that if u is a solution of (1)–(3), $f \in H$, then $\|u\|_{E,t}$ is finite, does not depend on t, and is equal to $\|f\|_E$. It is also known ([4]) that in this case

$$\lim_{t\to\infty} \|u\|_{E,t,R} = 0. \tag{4}$$

However, uniformity of this convergence to zero over the set of initial data $f \in H(R)$ is far from always attained, i.e., the validity, for u the solution of (1) –(3), of the inequality

$$\|u\|_{E,t,R} \le \varepsilon_R(t)\|f\|_E, \qquad f \in H(R), \qquad \varepsilon_R(t) \to 0, \quad t \to \infty, \tag{5}$$

where $\varepsilon_R(t)$ does not depend on f.

Examples of domains for which uniformity does not hold can be obtained from considerations of geometric optics. These are domains such that, for a given $S(R) \supset B$ and for arbitrarily large T, there exist a point $x \in S(R) \cap CB$ and a direction θ for which the ray emitted from x in the direction θ and reflected from ∂B according to the laws of geometric optics will remain in $S(R)$ for a time greater than T.

Although there is a conjecture that in all other cases uniformity holds, a positive result for the first boundary-value problem has been proved only for star-shaped B ([1]).

It is also known ([2]) that for any domain estimate (5) entails

$$\|u\|_{E,t,R} \le c_R \exp(-a_R t) \cdot \|f\|_E, \qquad a_R > 0; \tag{6}$$

a_R, c_R do not depend on f, t; u is the solution of (1)–(3) with initial data $f \in H(R)$.

In the present paper we introduce a broader class of domains than the star-shaped ones, and establish for them result (6).

2. **Definition.** A bounded open domain B with boundary of class C^1 is called **almost star-shaped** if there exist a D-bounded open neighborhood $\overline{B}$, a single-valued function $\varphi \in C^2(\overline{D} \cap CB)$, and a constant c_0 such that:

(I) $\varphi(x) < c_0,\ x \in D \cap CB,\ \varphi(x) = c_0,\ x \in \partial D.$

(II) $|\operatorname{grad} \varphi(x)| \ge \text{const} > 0,\ x \in \overline{D} \cap CB.$

(III) The level surfaces $\varphi(x) = c$ are strictly convex; the radius of curvature in all directions at all points of $CB \cap \overline{D}$ is uniformly bounded above.

(IV) At the points of intersection of the level surfaces with ∂B, their outer normals form with the outer normal to ∂B no more than a right angle.

Remark 1. If the level surfaces are spheres with a common center, then B is star-shaped, and conversely. Almost star-shaped domains are a natural generalization of star-shaped ones.

Remark 2. In the definition we do not require the connectedness of CB, since it follows from (I)–(IV).

3. **Theorem 1.** *If l is odd and greater than 1, and B is almost star-shaped with boundary of class C^2, then for any R there exist constants $a_R > 0, c_R$ such that for the solution of problem (1)–(3) with $f \in H(R)$ inequality (6) holds.*

Theorem 1 follows from Lemma 1 and from the result that estimate (5) entails estimate (6). Using the methods of semigroup theory developed in ([4]), one can establish that a stronger statement holds.

Theorem 2. *Under the assumptions of Theorem 1 there exist constants $c, \alpha > 0$ such that for all solutions of problem (1)–(3) with $f \in H(R_1)$*

$$\|u\|_{E,t,R_2} \leq c \exp(-\alpha t + \alpha R_1 + \alpha R_2)\|f\|_E \tag{7}$$

for any t, R_1, R_2.

4. **Lemma 1.** *Under the assumptions of Theorem 1, inequality (5) holds.*

Lemma 2. *If B is almost star-shaped, then for any R there exists a constant c such that*

$$\|u\|^2_{E,t,R} \leq \frac{c}{t}\|f\|^2_E + c \sup_{0\leq\tau\leq t} \|u(\tau)\|^2_{L_2(CB)}, \tag{8}$$

if u is the solution of problem (1)–(3), $f \in H(R)$, $t \geq 1$.

Lemma 3 (geometric). *If B is almost star-shaped, then there exist single-valued functions $\psi \in C^2(CB)$ and $\chi \in C^2(CB)$ such that:*

(I) $$\frac{1}{2} + \frac{1}{2}\Delta\psi(x) - \sum_{i,j=1}^{l} \psi_{x_i x_j}\xi_i\xi_j \leq \chi(x) \leq -\frac{1}{2} + \Delta\psi(x), \qquad x \in CB,$$

$$\xi \in R^l, \qquad |\xi| = 1.$$

(II) $\operatorname{grad}\psi(x) \cdot n(x) \geq 0$, $x \in \partial B$, $n(x)$ is the outer normal to ∂B at the point x.

(III) There exists a constant $\bar{c}$ such that

$$|\operatorname{grad}\psi(x)| \leq |x| - R + \bar{c}, \qquad |\chi(x)| \leq \bar{c}, \qquad |\Delta\chi(x)| \leq \bar{c}, \qquad x \in CB;$$

R is fixed.

We do not give the proof of Lemma 3 here. Instead of requirements (I)–(IV) in the definition of almost star-shapedness, one could have used assertions (I)–(III) of Lemma 3, but they seem unnatural to us from the geometric point of view.

The derivation of Lemma 2 from Lemma 3 can be carried out by multiplying equation (1) by $(2\bar{c}+t)\bar{u}_t + \sum_{i=1}^{l} \psi_{x_i}\bar{u}_{x_i} + \chi\bar{u}$, where $\psi, \chi, \bar{c}$ are obtained in Lemma 3. The resulting product is integrated over $x \in CB,\ 0 \le t \le t_1$; (2), (3), the properties (I)–(III) of $\psi, \chi, \bar{c}$, and also the fact that f belongs to $H(R)$, are used.

5. **Proof of Lemma 1.** Define the operator $S : H(R) \to H$ as follows: $h = Sf,\ h = \{h_0, h_1\}$, where $h_1 = f_0,\ h_0$ is the solution of $\Delta h_0 = f_1,\ x \in CB,\ h_0|_{\partial B} = 0,\ h \in H$ (in general, the membership $h \in H$ entails $h_0|_{\partial B} = 0$); $h \in H$ is defined uniquely for any $f \in H(R)$.

From the chain of inequalities

$$\sum_{i=1}^{l} \|h_0, x_i\|^2_{L_2(CB)} = -(h_0, \Delta h_0)_{CB} = -(h_0, f_1)_{CB} \le \|h_0\|_{L_2(S(R)\cap CB)} \|f_1\|_{L_2(CB)}$$

and from the definitions of $H,\ H(R),\ S$, it follows that S is bounded and completely continuous. Let v be the solution of (1)–(3) with initial data Sf; then $u = v_t,\ \|u(\tau)\|_{L_2(CB)} = \|v_\tau(\tau)\|_{L_2(CB)} \le \|Sf\|_E$, whence, and from equality (8), it follows that

$$\|u\|^2_{E,t,R} \le \frac{c}{t}\|f\|^2_E + c\|Sf\|^2_E \qquad (t \ge 1). \tag{9}$$

Now let $\varepsilon > 0$ be given. Then $H(R) = H^{(1)} \oplus H^{(2)}$, where, for every $f^{(1)} \in H^{(1)}$, the inequality $\|Sf\|^2_E \le \frac{\varepsilon}{4c}\|f\|_E$ holds, and $H^{(2)}$ is finite-dimensional. Then from (4) and the finite-dimensionality of $H^{(2)}$ there follows the existence of T' such that, for $t \ge T'$,

$$\|u^{(2)}\|^2_{E,t,R} \le \frac{\varepsilon}{2}\|f^{(2)}\|^2_E$$

for all $f^{(2)} \in H^{(2)}$ and $u^{(2)}$ the solution of problem (1)–(3) with initial data $f^{(2)}$. Let $f \in H(R)$, and let u be the solution of (1)–(3) with initial data f, $f = f^{(1)} + f^{(2)},\ f^{(k)} \in H^{(k)}$; then $u = u^{(1)} + u^{(2)}$, where $u^{(k)}$ is the solution of (1)–(3) with initial data $f^{(k)}$,

$$\|u\|^2_{E,t,R} \le 2(\|u^{(1)}\|^2_{E,t,R} + \|u^{(2)}\|^2_{E,t,R}) \le \varepsilon(\|f^{(1)}\|^2_E + \|f^{(2)}\|^2_E) = \varepsilon\|f\|^2_E$$

for $t \ge \max(1, T', 4c/\varepsilon)$. (We have used the definition of $H^{(k)}$ and the inequalities for $\|u^{(k)}\|_{E,t,R}$.) The lemma is proved.

6. There are examples showing that the class of almost star-shaped domains is very precise for the problem under consideration. Thus, domains of

the type of an open horseshoe (with diverging sides) are almost star-shaped (but, generally speaking, not star-shaped). The limiting domain—the horseshoe with parallel sides—is not almost star-shaped, and the main result is false for it. Star-shaped domains do not allow examples of this kind.

In conclusion, the author expresses his gratitude to M. D. Ramazanov for discussion.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR
Novosibirsk

Received
5 V 1969

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Note: Figure translations are in progress. See original paper for figures.