ON A CONDITION FOR REGULARITY OF A GAUSSIAN STATIONARY PROCESS

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Abstract

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ON A CONDITION FOR REGULARITY OF A GAUSSIAN STATIONARY PROCESS

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1. Let $\xi(t)$ be a stationary, in the narrow sense, process with discrete or continuous time. Denote by $\mathfrak{M}_a^b$ the σ-algebra of events generated by the course of the process on the interval $[a, b]$, $-\infty \le a < b \le \infty$. A. N. Kolmogorov proposed the following condition of weak dependence of events generated by the past and the future of the process (see ([1])):

$$\beta(\tau) = E\left\{\sup_{A\in\mathfrak{M}_{-\infty}^0} |P\{A \mid \mathfrak{M}_\tau^\infty\} - P\{A\}|\right\} \xrightarrow[\tau\to\infty]{} 0. \tag{1}$$

Denote by $P_{0,\tau}\{\cdot\}$ the measure induced by the process $\xi(t)$ on the σ-algebra $\mathfrak{M}_{-\infty}^0 \cup \mathfrak{M}_\tau^\infty$, and by $P_{1,\tau}\{\cdot\}$ the measure on the same σ-algebra defined, for $A \in \mathfrak{M}_{-\infty}^0$, $B \in \mathfrak{M}_\tau^\infty$, by the equality

$$P_{1,\tau}\{AB\} = P_{0,\tau}\{A\}P_{0,\tau}\{B\}.$$

V. A. Volkonskii and Yu. A. Rozanov ([2]) found that

$$\beta(\tau) = \tfrac{1}{2}\operatorname{Var}[P_{0,\tau} - P_{1,\tau}]. \tag{2}$$

Starting from (2), they obtained conditions, expressed in terms of the spectral density (s.d.), sufficient for a stationary Gaussian process with discrete time to satisfy condition (1). In the present note we give a scheme of proof of Theorem 1 formulated below, which makes it possible to obtain a complete description of stationary Gaussian processes with discrete time satisfying condition (1).

Theorem 1. *In order that a stationary Gaussian process with discrete time satisfy condition (1), it is necessary and sufficient that it have an s.d.* $f(\lambda)$ *representable in the form*

$$f(\lambda) = |P(e^{i\lambda})|^2 f_1(\lambda), \tag{3}$$

where:

1) *$P(z)$ is a polynomial with roots on the circle $|z| = 1$;*
2) $$\ln f_1(\lambda) \sim \sum_{-\infty}^{\infty} f_k e^{ik\lambda},$$

   where

$$\sum_{-\infty}^{\infty} |k|\,|f_k|^2 < \infty. \tag{4}$$

The proof of this theorem is contained in §§ 2, 3. There also some results are given for processes with continuous time. In § 4 results are given on the rate of decrease of $\beta(\tau)$.

2. We note that if a stationary, in the narrow sense, process satisfies condition (1), then it is regular in the sense of A. N. Kolmogorov (see ([1])) and, consequently, has an s.d. $f(\lambda)$ satisfying, in the case of discrete time,

$$\int_{-\pi}^{\pi} \ln f(\lambda)\, d\lambda > -\infty. \tag{5}$$

It follows from condition (5) (see (3)) that there exists an outer function $g(z) \in H^2$ inside the unit disk such that $|g(e^{i\lambda})|^2 = f(\lambda)$. Put

$$G(\lambda) = \frac{\overline{g(e^{i\lambda})}}{g(e^{i\lambda})} \sim \sum_{-\infty}^{\infty} g_k e^{ik\lambda}. \tag{6}$$

Theorem 2. *In order that a stationary Gaussian process with discrete time satisfy condition (1), it is necessary and sufficient that it have a spectral density $f(\lambda)$ satisfying condition (5), and that the function $G(\lambda)$, defined in (6), be such that*

$$\sum_{-\infty}^{0} |k|\,|g_k|^2 < \infty. \tag{7}$$

In the case of a process with continuous time, regularity implies the existence for the process of a spectral density $f(\lambda)$ satisfying the condition

$$\int_{-\infty}^{\infty} \frac{\ln f(\lambda)}{1+\lambda^2}\, d\lambda < \infty. \tag{5'}$$

It follows from condition (5′) (see (3)) that there exists an outer function $g(z) \in H^2$ in the right half-plane such that $|g(i\lambda)|^2 = f(\lambda)$. Put

$$G(\lambda) = \frac{\overline{g(i\lambda)}}{g(i\lambda)} \frac{1}{1 - i\lambda} \sim \int_{-\infty}^{\infty} \hat{G}(s) e^{is\lambda} \, ds. \tag{6'}$$

Theorem 2′. *In order that a Gaussian stationary process* (t) *with continuous time satisfy condition (1), it is necessary and sufficient that it have a spectral density* $f(\lambda)$ *satisfying condition (5′), and that the function* $G(\lambda)$*, defined in (6′), be such that*

$$\int_{-\infty}^{\infty} |\hat{G}(s)|^2 |s| \, ds < \infty. \tag{7}$$

We give the scheme of the proof of Theorem 2.

Necessity. From condition (1) it follows that the process $\xi(t)$ has a spectral density $f(\lambda)$ satisfying condition (5). Denote by $H_a^b(i)$, $i = 0, 1,\ -a, b \geq 0$, the Hilbert space of random variables $\eta(\omega)$ of the form

$$\eta = \sum_{t \in [a,b]} c(t) \xi(t)$$

with scalar product

$$(\eta_1, \eta_2) = \int_\Omega \eta_1(\omega) \overline{\eta_2(\omega)} \, P_{i,\theta}\{d\omega\}.$$

Consider the linear operator B_n from $H_0^n(0)$ into $H_0^n(1)$ such that $(x, y)_0 = (B_{nx}, y)_1,\ x, y \in H_0^n(i)$. We calculate $\|B_n - E\|$–the Hilbert-Schmidt norm of the operator $B_n - E$. If T is a linear isometry of the space $H_0^n(0)$ into $L_2(-\pi, \pi)$ such that $T\xi(t) = e^{it\lambda} g(\lambda)$, then, by Beurling' s theorem ((3), p. 101), we have $TH_{-\infty}^n(i) = e^{in\lambda} H^2$. It is easy to see that $TH_0^\infty = \frac{g}{\bar{g}} H^2$. Hence

$$\|B_n - E\|^2 = \sum_k |g_{k+n}|^2 k. \tag{8}$$

From condition (1) follows the mutual absolute continuity of the Gaussian measures $P_{0,n}$, $P_{1,n}$ for sufficiently large n. Hence, using one

Feldman's result (see ([4])), we infer that $\|B_n - E\| < \infty$. From the last inequality and relation (8), (7) follows.

Sufficiency. Relation (8) implies the mutual absolute continuity of the measures mentioned above, if $H_0^\infty(0) \cap H_{-\infty}^n = \{0\}$. Therefore, from relations (7), (8), and Theorem 4.1 of ([2]), relation (1) follows at once.

3. We finish the proof of **Theorem 1**.

Sufficiency. It is verified directly that (7) follows from (3).

Necessity. From condition (1) it follows that

$$\alpha(\tau) = \sup_{A\in\mathfrak{M}^{0}_{-\infty},\ B\in\mathfrak{M}^{\infty}_{\tau}} |P\{AB\} - P\{A\}P\{B\}| \xrightarrow[\tau\to\infty]{} 0. \tag{9}$$

Denote by $L_2(f)$ the Hilbert space of functions square-integrable with weight f, with scalar product

$$(\varphi, \psi)_f = \int_{-\pi}^{\pi} \varphi(\lambda)\overline{\psi(\lambda)} f(\lambda)\, d\lambda.$$

By a theorem of A. N. Kolmogorov and Yu. A. Rozanov ([5]), condition (9) is equivalent to the condition

$$\rho(\tau; f) = \sup_{\varphi,\psi} \left|(e^{i\tau\lambda}\varphi, \overline{\psi})_f\right| \xrightarrow[\tau\to\infty]{} 0,$$

where the supremum is taken over all polynomials $\varphi, \psi \in H^2$ with norms $\|\varphi\|_f = \|\psi\|_f = 1$. Hence, on the basis of the results of papers ([6],[7]), it follows that the function $f(\lambda)$ can be written in the form $|P(e^{i\lambda})|^2 f_1(\lambda)$, where $P(z)$ is a polynomial with roots on the circle $|z| = 1$, $\rho(1; 1/f_1) = \rho < 1$. It is not difficult to see that $\overline{P}/P = ae^{-ik\lambda}$, $|a| = 1$, k is the degree of P, so that together with the function $g(z)$ satisfying condition (7), the outer function $g_1(z)$, where $|g_1(e^{i\lambda})|^2 = f_1(\lambda)$, satisfies condition (7). The latter condition can be written in the form

$$\sum \inf_A \left\| \frac{\overline{g_1}}{g_1} - e^{-in\lambda} A \right\|_2^2 < \infty, \tag{10}$$

where $\|\cdot\|_2$ is the norm in $L_2(-\pi, \pi)$, and the infimum is taken over all $A \in H^2$. Choose a sequence of polynomials $A_n \in H^2$ for which the series (10) converges. Put $e^{-in\lambda} g_1 A_n = Q_n(e^{-in\lambda}) + B_n$, where Q_n is a polynomial of degree n, and all $B_n \in H^2$. Then

$$\left\|\overline{g_1}/g_1 - e^{-in\lambda} A_n\right\|^2 = \|(g_1 - Q_n) - B_n\|_{1/f_1}^2 \geq (1-\rho) \left\|g - \overline{Q}_n\right\|_{1/f_1}^2 .$$

Consequently, the series

$$\sum \inf_{P_n} \left\|g_1 - \overline{P}_n\right\|_{1/f_1}^2 , \tag{11}$$

converges, where the infimum is taken over all polynomials $P_n(e^{i\lambda})$ of degree not exceeding n.

Introduce the polynomials $\varphi_\nu(z)$, orthonormal with weight $1/f_1$. From the properties of such polynomials ([8]) one can infer that convergence of the series (11) is equivalent to the inequality

$$\sum_1^\infty \nu\,|\varphi_\nu(0)|^2 < \infty.$$

Let $\varphi_\nu^*(z) = z^\nu \overline{\varphi_\nu(1/z)}$. The polynomials $\varphi_{\nu,n}(z)$, orthonormal with weight $|\varphi_n^*(e^{i\lambda})|^{-2}$, have the form $\varphi_{\nu,n}(z) = \varphi_\nu(z)$, $\nu \le n$; $\varphi_{\nu,n}(z) = z^{\nu-n}\varphi_n(z)$, $\nu \ge n$. Consequently,

$$\sum_{\nu=1}^\infty \nu\,|\varphi_{\nu,n}(0)|^2 \le \sum_{\nu=1}^\infty \nu\,|\varphi_\nu(0)|^2. \tag{12}$$

Denote by $D_{\nu,n}$ the ν-th Toeplitz determinant constructed with respect to the weight $|\varphi_n^*|^{-2}$. Let

$$G(h) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln h(\lambda)\,d\lambda\right\}.$$

By a theorem of G. Szegő ([8], p. 101),

$$\lim_{\nu\to\infty} \frac{D_{\nu,n}}{\left(G\left(|\varphi_n^*|^{-2}\right)\right)^{\nu+1}} = \exp\left\{\frac{1}{\pi}\iint_{|z|\le 1}\left|\frac{\varphi_n^{*\prime}(z)}{\varphi_n^*(z)}\right|^2 d\sigma\right\}.$$

From this and from (12) it follows that all the integrals on the right-hand side of (13) are uniformly bounded. As $n \to \infty$, $\varphi_n^*(z) \to g_1(z)$ uniformly in every disk $|z| \le r < 1$. Therefore

$$\sum_{-\infty}^{\infty} |k|\,|f_k|^2 = \frac{2}{\pi}\iint_{|z|\le 1}\left|\frac{g_1'(z)}{g_1(z)}\right|^2 d\sigma = \lim_{n\to\infty}\frac{2}{\pi}\iint_{|z|\le 1}\left|\frac{\varphi_n^{*\prime}(z)}{\varphi_n^*(z)}\right|^2 d\sigma < \infty.$$

4. In the case of continuous time, one can give conditions sufficient for condition (1) to hold.

Theorem 3. Let the spectral density $f(\lambda)$ of a stationary Gaussian process with continuous time be represented in the form $|B(i\lambda)|^2 f_1(\lambda)$, where $B(z)$ is an entire function of finite degree, and the function $f_1(\lambda)$ is such that: 1) $0 < m \le f_1(\lambda) \le M$; 2) the function $\hat{f}(\lambda)$—the Fourier transform of $f_1(\lambda)/(1+\lambda^2)$—satisfies the inequality

$$\int_{-\infty}^{\infty} |\lambda|\, |\hat{f}(\lambda)|^2 \, d\lambda < \infty.$$

Then the process $\xi(t)$ satisfies condition (1).

In the case of a process with discrete time, one can, just as was done in paper ([9]) (see Theorems 3, 4, 5), completely describe the class of those spectral densities for which $\beta(n)$ decreases sufficiently rapidly. We give one such theorem.

Theorem 4. In order that $\beta(n) = O\left(n^{1/2-r-\alpha}\right)$, where r is an integer, $r+\alpha > 1/2$, $0 < \alpha < 1$, it is necessary and sufficient that the spectral density $f(\lambda)$ be representable in the form $|P(e^{i\lambda})|^2 f_1(\lambda)$, where $P(e^{i\lambda})$ is a trigonometric polynomial, and the function $f_1(\lambda)$ is positive, $f_1(\lambda) \geq m > 0$, and has an r-th derivative satisfying a Hölder condition of order α.

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Note: Figure translations are in progress. See original paper for figures.