EMBEDDING THEOREMS FOR WEIGHTED SPACES OF FRACTIONAL ORDER WITH MIXED NORM

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.947.42+513.881

MATHEMATICS

R. D. KULOV

EMBEDDING THEOREMS FOR WEIGHTED SPACES OF FRACTIONAL ORDER WITH MIXED NORM

(Presented by Academician S. L. Sobolev on 11 III 1969)

Embedding theorems of the type $M \to N^*$, where M is an ordinary functional space $\left(W_p^l, W_{p,\alpha}^l, B_p^r, H_p^r, S_p^{rH}\right)$, and N is a space with mixed norm, have by now been studied rather well in a number of works ([6,8,10,11,14]). As for embedding theorems of the type $N \to N$, only for the spaces W_p^l (l integer), H_p^r, S_p^{rH} are there several results ([9,12,13]). In the present note this gap is filled for weighted spaces of fractional order $W_{(\mathbf{p})\alpha}^l(\dot{E}^n)$ with mixed norm. The results obtained are definitive.

Let E^n be the n-dimensional material Euclidean space of points $x = (x_1, \dots, x_n)$; E^m an m-dimensional subspace of the space E^n;

$$\dot{E}^n = \{x : x_n > 0\}, \quad \dot{E}^{n-m} = \dot{E}^n \setminus E^m, \quad \dot{E}^n = E^{n_1} \times \dots \times E^{n_k}, \quad E^m = E^{m_1} \times \cdots$$

$$\cdots \times E^{m_\tau},$$

where the decompositions of $\dot{E}^n$ and E^m into subspaces are independent of one another;

$$E^{n_\nu} \cap E^{m_j} = E^{m_{\nu j}}; \quad \dot{E}^{n-m} \cap E^{n_\nu} = E^{n_\nu} \left(\sum_{j=1}^{\tau} m_{\nu j} + n'_\nu = n_\nu \right); \quad \mathbf{p} = (p_1, \dots, p_k), \quad \mathbf{q} = (q_1, \dots, q_\tau),$$

$$\mathbf{l} = (l_1, \dots, l_n), \quad x' = (x_1, \dots, x_{n-1}, 0) = (x'_{n_1}, \dots, x'_{n_k}), \quad x'_{n-m} = (x_{m+1}, \dots, x_{n-1}, 0) = (x'_{n'_1}, \dots, x'_{n'_k})$$

(where x_{n_ν} and $x'_{n'_\nu}$ belong respectively to E^{n_ν} and $E^{n'_\nu}$, $\nu = 1, \dots, k$);

$$\int_{x_1}^{x_1+h\sigma_1} dy_1 \cdots \int_{x_{n-1}}^{x_{n-1}+h\sigma_{n-1}} dy_{n-1} \int_0^{h\sigma_n} [\cdot]\, dy_n = \int_{\bar{x}}^{x'+h\bar{\sigma}} [\cdot]\, dy,$$

(h is an arbitrary positive parameter);

$$\left(\int_{x_{n_k}}^{x'_{n_k}+h\bar{\sigma}_{n_k}} dy_{n_k} \cdots \left(\int_{x_{n_1}}^{x'_{n_1}+h\bar{\sigma}_{n_1}} [\cdot]^{p_1}\, dy_{n_1} \right)^{p_2/p_1} \cdots \right)^{1/p_k} = A^{(p_k;1)}_{(x_{n_k};1;h)\bar{\sigma}_{n_k};1}[\cdot].$$

Further, let $f(y)$ be some smooth finite function in E^n and

$$\|f\|_{L^{l_i}_{(\mathbf{p}),\alpha}(\dot{E}^n)} = \left(\int_0^\infty \frac{dt}{t^{1+p_k l_i}} \left\| y_n^{\alpha/p_1} \Delta_i(t) f(y) \right\|^{p_k}_{L_{(\mathbf{p})}(\dot{E}^n)} \right)^{1/p_k} < \infty$$

(where $L_{(\mathbf{p})}(\dot{E}^n)$ is the space with mixed norm; $\Delta_i(t)f(y)$ is the finite difference of first order with step t in the variable x_i);

$$\|f\|_{L^{l}_{(\mathbf{p}),\alpha}(\dot{E}^n)} = \sum_{i=1}^{n} \|f\|_{L^{l_i}_{(\mathbf{p}),\alpha}(\dot{E}^n)},$$

$$\|f\|_{W^{l}_{(\mathbf{p}),\alpha}(\dot{E}^n)} = \|f\|_{L_p \dot{E}^n} + \|f\|_{L^{l}_{(\mathbf{p}),\alpha}(\dot{E}^n)}.$$

---

* $E_1 \to E_2$ denotes the set-theoretic embedding of the class E_1 in the class E_2 with the inequality $\|f\|_{E_2} \le c\|f\|_{E_1}$ satisfied.

We shall call $L_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n)$ and $W^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n)$ the closures of the set of smooth finite functions in the corresponding norms.

Theorem 1. If $1 < p_\nu \le q_j < \infty$ $(\nu = 1, \dots, k;\ j = 1, \dots, \tau)$, $1 \le m < n$,

$$-\frac{p_1}{\max_\nu p_\nu} < \alpha < \frac{p_1}{\max_\nu p'_\nu}, \quad 0 < l_i < 1 \quad \left(\sigma_i = \frac{1}{l_i};\ i = 1, \dots, n \right),$$

$$\varepsilon = 1 - \sum_{i=1}^{n} \sigma_i - \frac{\alpha\sigma_n}{p_1} + \sum_{\nu=1}^{k} \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^{\tau} \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma^{(j)}_\beta > 0$$

(where $\sigma_{\nu\beta}$, $\sigma^{(j)}_\beta$ are numbers corresponding to the coordinate axes of the subspaces E^{n_ν} and E^{m_j}), $f \in W^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n)$, then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the relation

$$W^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n) \to L_{(\mathbf{q})}(E^m)$$

holds.

Theorem 2. If $1 < p_\nu \le q_j < \infty$ $(\nu = 1, \dots, k;\ j = 1, \dots, \tau)$, $1 \le m < n$,

$$-\frac{p_1}{\max_\nu p_\nu} < \alpha < \frac{p_1}{\max_\nu p'_\nu}, \quad 0 < l_i < 1 \quad \left(\sigma_i = \frac{1}{l_i};\ i = 1, \dots, n\right),$$

$$\varepsilon \ge \rho_s \sigma_s \quad (0 < \rho_s < 1;\ s = 1, \dots, m),$$

$$\varepsilon = 1 - \sum_{i=1}^n \sigma_i - \frac{\alpha\sigma_n}{p_1} + \sum_{\nu=1}^k \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} > 0, \quad f \in W^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n),$$

then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the relation

$$W^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n) \to \overline{L}^\rho_{(\mathbf{q})}(E^m)$$

holds.

From Theorems 1 and 2 follows the validity of the following theorem.

Theorem 3. If $1 < p_\nu \le q_j < \infty$ $(\nu = 1, \dots, k;\ j = 1, \dots, \tau)$, $1 \le m < n$,

$$-\frac{p_1}{\max_\nu p_\nu} < \alpha < \frac{p_1}{\max_\nu p'_\nu}, \quad 0 < l_i < 1 \quad \left(\sigma_i = \frac{1}{l_i};\ i = 1, \dots, n\right),$$

$$\varepsilon = 1 - \sum_{i=1}^n \sigma_i - \frac{\alpha\sigma_n}{p_1} + \sum_{\nu=1}^k \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^\tau \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} > 0,$$

$$\varepsilon \ge \rho_s \sigma_s \quad (0 < \rho_s < 1;\ s = 1, \dots, m), \quad f \in W^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n),$$

then for any fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the relation

$$W^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n) \to \overline{W}^\rho_{(\mathbf{q})}(\overset{+}{E}{}^m)$$

holds.

Theorem 4. If

$$1 < p_k \le p_{k-1} \le \dots \le p_1 < q_j < \infty \quad (j = 1, \dots, \tau),$$

$$-1 < \alpha < \frac{p_1}{p'_k}, \quad 1 \le m < n, \quad 0 < l_i < 1 \quad (\sigma_i = 1/l_i;\ i = 1, \dots, n),$$

$$1 - \sum_{i=1}^{n} \sigma_i - \frac{\alpha\sigma_n}{p_1} + \sum_{\nu=1}^{k} \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^{\tau} \sum_{\nu=1}^{k} \left(\frac{1}{p_\nu} + \frac{1}{q_j}\right) \sum_{\beta=1}^{m_{\nu j}} \sigma_\beta^{(\nu j)} = 0$$

(where $\sigma_{\nu\beta}$, $\sigma_\beta^{(\nu j)}$ are numbers corresponding to the coordinate axes of the subspaces E^{n_ν} and $E^{m_{\nu j}}$), $f \in L^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n)$, then for fixed $x_{m+1}, \dots, x_{n-1}$ and $x_n = 0$ the relation

$$L^l_{(\mathbf{p}),\alpha}(\overset{+}{E}{}^n) \to L_{(\mathbf{q})}(E^m)$$

holds.

We outline the scheme of the proof, for example, of Theorem 2. From the integral identity of V. P. Il' in ([3]) we have

$$\|\Delta_s(t) f(x')\|_{L_{(q)}(E^m)} \ll \|\Delta_s(t) F_1(x', h)\|_{L_{(q)}(E^m)} + \|\Delta_s(t) F_2(x', t, h)\|_{L_{(q)}(E^m)} + \|\Delta_s(t) F_3(x', t, h)\|_{L_{(q)}(E^m)}, \tag{1}$$

where

$$F_1(x', h) = \frac{c}{h^\omega} \int_0^{h\bar{\sigma}} f(x' + y) \Pi(y, h)\, dy,$$

$$F_2(x', t, h) = \sum_{i=1}^{n} C_i \int_0^{t^{1/\sigma_s}} \frac{d\vartheta}{\vartheta^{1+\omega}} \int_0^{\vartheta\bar{\sigma}} dy \int_0^{\vartheta^{\sigma_i} - y_i} \Delta_i(\chi) f(x' + y) R_i(y, \chi, \vartheta)\, d\chi,$$

$$F_3(x', t, h) = \sum_{i=1}^{n} C_i \int_{t^{1/\sigma_s}}^{h} \frac{d\vartheta}{\vartheta^{1+\omega}} \int_0^{\vartheta\bar{\sigma}} dy \int_0^{\vartheta^{\sigma_i} - y_i} \Delta_i(\chi) f(x' + y) R_i(y, \chi, \vartheta)\, d\chi;$$

each term of the right-hand side of inequality (1) admits the estimate

$$\|\Delta_s(t) F_1(x', h)\|_{L_{(q)}(E^m)} \ll t \|D_{sF} 1(x_1, \dots, x_s, \dots, x_{n-1}, 0; h)\|_{L_{(q)}(E^m)} \ll$$

$$\ll cth^{\sum_{\nu=1}^{k}{}_{\nu}^{\frac{1}{p}} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta}+\sum_{j=1}^{\tau} \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)}-\sum_{j=1}^{n} \sigma_j-\sigma_s} \|f\|_{L_{(p)}(\dot{E}^n)}; \tag{2}$$

$$\|\Delta_s(t)F_3(x',t,h)\|_{L_{(q)}(E^m)} \ll t\|D_{sF}3(x_1,\dots,x'_s,\dots,x_{n-1},0;t,h)\|_{L_{(q)}(E^m)} \ll$$

$$\ll t\sum_{i=1}^{n} C_i \left\| \int_{t^{1/\sigma_s}}^{h} \frac{d\vartheta}{\vartheta^{1+\lambda_i-\varepsilon+\sigma_i-\alpha\sigma_n/p_1+\sigma_i\gamma}} \int_0^{\vartheta^{\sigma_i}} \chi^{\gamma-(1/p_k+l_i)}\,d\chi \int_{x'}^{x'+\vartheta\bar{\sigma}} \Delta_i(\chi)f(y)\,dy \right\|_{L_{(q)}(E^m)}; \tag{3}$$

$$\|\Delta_s(t)F_2(x',t,h)\|_{L_{(q)}(E^m)} \ll C \left\| \int_0^{t^{1/\sigma_s}} \frac{d\vartheta}{\vartheta^{1+\lambda_i-\varepsilon+\gamma\sigma_i-\alpha\sigma_n/p_1}} \times \right.$$

$$\left. \times \int_0^{\vartheta^{\sigma_i}} \chi^{\gamma-(1/p_k+l_i)}\,d\chi \int_{x'}^{x'+\vartheta\bar{\sigma}} \Delta_i(\chi)f(y)\,dy \right\|_{L_{(q)}(E^m)} \tag{4}$$

(in inequalities (3), (4), γ and λ_i are determined from $0 < \gamma \le 1/p_k + l_i$,

$$\lambda_i = \frac{\sigma_i}{p_k} + \sum_{\nu=1}^{k} \frac{1}{p_\nu} \sum_{\beta=1}^{n_\nu} \sigma_{\nu\beta} + \sum_{j=1}^{\tau} \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} \Big).$$

Using the definition of the space $L_{(q)}^{\rho_s}$, we obtain

$$\|f\|_{L_{(q)}^{\rho_s}(E^m)} \ll \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\rho_s q_\tau}} \|\Delta_s(t)f(x')\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau} +$$

$$+ \left(\int_{h^{\sigma_s}}^{\infty} \frac{dt}{t^{1+\rho_s q_\tau}} \|\Delta_s(t)f(x')\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau} \equiv A[0,h^{\sigma_s}] + A[h^{\sigma_s},\infty].$$

Taking inequalities (1) into account, for $A[0,h^{\sigma_s}]$ we obtain the estimate

$$A[0,h^{\sigma_s}] \ll \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+q_\tau\rho_s}} \|\Delta_s(t)F_1(x',h)\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau} +$$

$$+ \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+p_s q_\tau}} \|\Delta_s(t)F_2(x',t,h)\|_{L_{(q)}(E^m)}^{q_\tau} \right)^{1/q_\tau}$$

$$+\left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+p_s q_\tau}} \|\Delta_s(t) F_3(x',t,h)\|^{q_\tau}_{L_{(q)}(E^m)}\right)^{1/q_\tau} \equiv A_1^{(s)} + A_2^{(s)} + A_3^{(s)}.$$

We estimate $A_1^{(s)}, A_2^{(s)}, A_3^{(s)}$: on the basis of inequality (2)

$$A_1^{(s)} \ll ch^{\sum_{\nu=1}{}^{k} \frac{1}{p'_\nu} \sum_{\beta=1}^{n_\nu} \sigma_\nu \beta + \sum_{j=1}^{\tau} \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} - \sum_{j=1}^{n} \sigma_j - \rho_s \sigma_s} \|f\|_{L_{(p)}(E^n)}, \qquad (5)$$

taking into account inequality (4),

$$A_2^{(s)} \ll ch^{\varepsilon-\rho_s\sigma_s} \sum_{i=1}^{n} \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\nu\sigma_i q_\tau/\sigma_s - \xi q_\tau/\sigma_s}} \left\| \left(\int_0^{t^{1/\sigma_s}} \frac{d\vartheta}{\vartheta^{1+p_k \sum_{j=1}{}^{\tau} \frac{1}{q_j} \sum_{\beta=1}^{m_j} \sigma_\beta^{(j)} + \xi p_k}}\right.\right.\right.$$

$$\left.\left.\left.\times \int_0^{\vartheta^{\sigma_i}} \chi^{\gamma p_k - (1+p_k l_i)}\, d\chi \left(A^{(p_k;1)}_{\left(x_{n_k;1};\, \vartheta^{\bar\sigma_{n_k;1}}\right)} \left[y_n^{a/p_1} \Delta_i(\chi) f(y)\right]\right)^{p_k}\right)^{1/p_k} \right\|^{q_\tau}_{L_{(q)}(E^m)}\right)^{1/q_\tau}$$

(where ξ is, for the time being, an arbitrary positive number).
Having made simple estimates for the norm in the curly bracket, we obtain

$$A_2^{(s)} \ll ch^{\varepsilon-\rho_s\sigma_s} \|f\|_{L^1_{(p),\alpha}(E^n)}. \qquad (6)$$

Similarly we obtain

$$A_3^{(s)} \ll ch^{\varepsilon-\rho_s\sigma_s} \|f\|_{L^1_{(p),\alpha}(E^n)}. \qquad (7)$$

It is easy to note that

$$A\left[h^{\sigma_s}, \infty\right] \ll ch^{-\sigma_s\rho_s} \|f\|_{\mathring{W}^{l}_{(p),\alpha}(E^n)}. \qquad (8)$$

From the estimates given above, taking into account inequalities (5), (6), (7), (8) and setting $h = 1$, we obtain the assertion of the theorem.

Taking this opportunity, I express my deep gratitude to Acad. S. L. Sobolev for his attention to this work, and also to A. Kh. Gudiev for valuable advice.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR
Novosibirsk

Received
11 III 1969

REFERENCES

1. S. L. Sobolev, *Matem. sborn.*, **4** (46), 3 (1938).
2. S. L. Sobolev, *Some applications of functional analysis in mathematical physics*, L., 1950.
3. V. P. Il' in, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **66** (1962).
4. L. N. Slobodetskii, *DAN*, **118**, No. 2 (1958).
5. A. Kh. Gudiev, *Differential Equations*, **2**, No. 2 (1966).
6. S. M. Nikol' skii, *Siberian Mathematical Journal*, **3**, No. 6 (1962).
7. A. D. Dzhabrailov, Candidate' s dissertation, Moscow, 1966.
8. A. Kh. Gudiev, *DAN*, **160**, No. 2 (1965).
9. A. Kh. Gudiev, *Izv. AN UzSSR*, No. 4 (1968).
10. A. D. Beriev, Candidate' s dissertation, Novosibirsk, 1968.
11. A. P. Uninskii, Candidate' s dissertation, Tomsk, 1967.
12. A. S. Dzhafarov, *Uch. zap. Azerbaijan State Univ. named after S. M. Kirov*, No. 2 (1963).
13. A. S. Dzhafarov, *Dokl. AzSSR*, **21**, No. 2 (1965).
14. Ya. S. Bugrov, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **77** (1965).
15. T. I. Amanov, Doctoral dissertation, Novosibirsk, 1967.

Note: Figure translations are in progress. See original paper for figures.