PHRAGMÉN-LINDELÖF TYPE THEOREMS FOR SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS IN AN INFINITE SLAB

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.944

MATHEMATICS

V. M. BOROK

PHRAGMÉN-LINDELÖF TYPE THEOREMS FOR SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS IN AN INFINITE SLAB

(Presented by Academician I. G. Petrovsky on 25 IV 1968)

We shall consider solutions of equations of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} + P\left(\frac{\partial}{\partial x}\right)\frac{\partial u(x,t)}{\partial t} + Q\left(\frac{\partial}{\partial x}\right)u(x,t) = 0 \tag{1}$$

in the slab $\Pi = \{(x,t) : 0 < t < T,\ x \in R^n\}$, $P\left(\frac{\partial}{\partial x}\right)$, $Q\left(\frac{\partial}{\partial x}\right)$ are polynomials in

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$$

with constant (complex) coefficients. The questions of existence and uniqueness of a solution of equation (1) under the boundary conditions

$$u(x,0) = u_0(x), \qquad u(x,T) = u_T(x), \qquad x \in R^n \tag{2}$$

were studied by us in papers ([1,2]). In ([2]) a class of equations of the form (1), called well-posed, was singled out. In the present paper, for solutions of well-posed equations (1) we establish certain Phragmén–Lindelöf type theorems: from a certain a priori estimate of the solution in Π and another, considerably stronger, estimate of the solution on the boundary Π' of the slab Π (for $t = 0$ and $t = T,\ x \in R^n$) it is concluded that the solution satisfies in Π estimates of the same type as on the boundary Π.

1°. Denote

$$D(s) \equiv \frac{1}{4}P^2(-is) - Q(-is), \qquad \Lambda(s) \equiv |\operatorname{Re} P(is)| - 2|\operatorname{Re}\sqrt{D(-s)}|,$$

$s = \sigma + i\tau,\ \sigma \in R^n,\ \tau \in R^n.$

Equation (1) is called **well-posed** ([2]) if, for $s = \sigma \in R^n$: 1) $D(\sigma)$ does not take the values $-k^2\pi^2T^{-2}$, $k = 1, 2, ...$; 2) there exists a constant L such that $\Lambda(\sigma) \leq L$.

Theorem 1. Let $u(x, t)$ be a solution of the well-posed equation (1) in Π. Then for every $l \geq 0$ there exist $l_1 \geq 0$ ($l_1 \geq l$) and $A > 0$ such that, if the following conditions are fulfilled: 1) the solution $u(x, t)$, together with its derivatives up to order l_1 (with respect to x), is continuous in $\Pi + \Pi'$ and satisfies the estimates

$$|D^k u(x, 0)| \leq C, \qquad |D^k u(x, T)| \leq C,$$

$$D^k = \partial^k / \partial x_1^{k_1} \dots \partial x_n^{k_n}, \qquad k = k_1 + \dots + k_n \leq l_1;$$

2) for $(x, t) \in \Pi$

$$|u(x, t)| \leq C_1 \exp\{C_2 |x|^\alpha\} \tag{3}$$

for some $C_1 > 0$, $C_2 > 0$, $0 < \alpha < 1$, then the inequalities

$$|D_x^k u(x, t)| \leq AC, \qquad (x, t) \in \Pi, \qquad 0 \leq k \leq l$$

hold.

Theorem 2. Let $u(x, t)$ be a solution of the well-posed equation (1) in Π. Then for any $l \geq 0$ and m ($-\infty < m < \infty$) there exist $l_1 \geq 0$ ($l_1 \geq l$) and $A > 0$ such that, if the following conditions are fulfilled: 1) the solution $u(x, t)$, together with its derivatives with respect to x up to order l_1, is continuous in $\Pi + \Pi'$ and satisfies the estimates

$$|D^k u(x, 0)| \leq C(1 + |x|)^m, \qquad |D^k u(x, T)| \leq C(1 + |x|)^m, \qquad 0 \leq k \leq l_1;$$

2) in Π estimate (3) holds, then the inequalities

$$|D_x^k u(x, t)| \leq AC(1 + |x|)^m, \qquad (x, t) \in \Pi, \quad 0 \leq k \leq l. \tag{4}$$

Theorem 3. Let (1) be a correct equation, and let $u(x, t)$ be its solution in Π. For any $l \geq 0$ there exists $l_1 \geq 0$ ($l_1 \geq l$) such that, if $u(x, t)$ satisfies the conditions: 1) $u(x, t)$ is continuous, together with its derivatives up to order l_1 (with respect to x), in $\Pi + \Pi'$, and the functions $D^k u(x, 0)$, $D^k u(x, T)$, $0 \leq k \leq l_1$, belong to the space $L_2(R^n)$; 2) estimate (3) holds in Π; then, for each $t \in (0, T)$, the function $u(x, t)$ and its derivatives up to order l are also square-integrable (over R^n). Moreover, the norm (in $L_2(R^n)$) of the function $u(x, t)$ and of its derivatives with respect to x up to order l is proportional (with a certain fixed constant $A = A(l)$) to the maximum of the norms (in $L_2(R^n)$) of the functions $u(x,0)$ and $u(x,T)$ and their derivatives up to order l_1.

Remark 1. We note that if in (3) one replaces α by 1, and C_2 by an arbitrary $\varepsilon > 0$, then none of Theorems 1, 2, 3 is, generally speaking, true. Let us denote

$$Z = \{s = \sigma + i\tau : \ D(s) = -k^2\pi^2T^{-2}, \quad k = 1, 2, ...\},$$

$$a = \inf_{s \in Z} |\operatorname{Im} s|.$$

Remark 2. If it is assumed that for the correct equation under consideration $0 < a < \infty$, then condition (3) in all three theorems can be weakened by replacing α by 1 and C_2 by C_3, $0 < C_3 < a$.

Remark 3. If $a = \infty$, then condition (3) can be weakened still further, by taking in it arbitrary $\alpha > 0$. In this case the results of Theorems 1, 2, 3 remain valid.

2°. A correct equation (1) is called an **equation of zero order** (2) if $a > 0$ and there exists a strip $|\tau| \leq b$ $(0 < b \leq a)$ in which $A(s) \leq L_1$ (where L_1 is some constant). The number $b > 0$ will be called the **exponent** of the equation.

For solutions of correct equations of zero order the following two theorems hold.

Theorem 4. Let (1) be a correct equation of zero order with exponent b, and let $u(x,t)$ be a solution of this equation in Π. Then for any $l \geq 0$ there exist $l_1 \geq 0$ and $A > 0$ such that, if the following conditions are fulfilled: 1) the solution $u(x,t)$ is continuous, together with its derivatives up to order l_1, in $\Pi + \Pi'$ and satisfies the estimates

$$|D^k u(x,0)| \leq C \exp\{\beta|x|\}, \qquad |D^k u(x,T)| \leq C \exp\{\beta|x|\}, \tag{5}$$

$$0 \leq k \leq l_1,$$

where β is any number such that $|\beta| < b$;

2)
$$|u(x,t)| \leq C_1 \exp\{C_2|x|\}, \qquad C_1 > 0, \quad \beta \leq C_2 < a, \tag{6}$$

then the solution $u(x,t)$ satisfies in Π the estimates

$$|D_x^k u(x,t)| \leq AC \exp\{\beta|x|\}, \qquad 0 \leq k \leq l.$$

Theorem 5. Suppose the conditions of the preceding theorem are fulfilled, but instead of estimate (5) the estimate

$$|D^k u(x,0)| \leq C \exp\{\beta|x|^\alpha\}, \qquad |D^k u(x,T)| \leq C \exp\{\beta|x|^\alpha\},$$

$$0 \leq k \leq l_1,$$

holds, where β is any fixed number (positive or negative), $0 < \alpha < 1$. Then, for the solution $u(x,t)$ in Π, the estimates

$$|D_x^k u(x,t)| \leq AC \exp\{\beta |x|^\alpha\}, \qquad 0 \leq k \leq l.$$

Remark 4. If $a = \infty$, then Theorems 4 and 5 remain valid if condition (6) is substantially weakened by replacing it with an estimate of the form (3), where $C_2 > 0$ is arbitrary, $\alpha > 1$ is arbitrary. If, however, $0 < a < \infty$, then, generally speaking, in (6) one cannot replace C_2 by a constant C_3, $C_3 > a$ (cf. Remarks 1 and 3).

3°. A correct equation (1) is called **strongly correct** (2) if there exist $L_1 > 0$, $h > 0$, and L_2 such that for $s = \sigma$

$$\Lambda(\sigma) \leq -L_1 |\sigma|^h + L_2;$$

it is called an **equation of zero genus** (with exponent b) if: 1) $a > 0$, and 2) in the strip $|\tau| \leq b$ $(0 < b \leq a)$ the estimate

$$\Lambda(s) \leq -L_{h'} |s|^{h'} + L_2'$$

holds with some $L_{h'} > 0$ and any $h' \in (0,h)$.

For strongly correct equations, Theorems 1, 2, 3 are valid for $l_1 = 0$ (i.e., estimates of the functions $u(x,0)$ and $u(x,T)$ themselves, without derivatives, and condition 2) of each of the theorems imply the corresponding estimate in Π of the solution $u(x,t)$ and its derivatives). For strongly correct equations of zero genus, the same is true with respect to Theorems 4 and 5. The remarks made above remain in force in this case as well.

Moreover, for solutions of strongly correct equations the following is valid.

Theorem 6. Let $u(x,t)$ be a solution of the strongly correct equation (1). Then there exists a constant $\gamma > 0$ (determined by the equation itself) such that for any $l \geq 0$ one can specify $A = A(l)$ such that, if: 1) the solution $u(x,t)$ is continuous in $\Pi + \Pi'$, and

$$|u(x,0)| \leq C \exp\{B|x|^\beta\}, \qquad |u(x,T)| \leq C \exp\{B|x|^\beta\},$$

where B is arbitrary, $0 < \beta < h/(h+\gamma)$; 2) for $(x,t) \in \Pi$ the estimate (3) holds with some a, $\beta \leq \alpha < 1$, then for the solution $u(x,t)$ the estimate

$$|D_x^k u(x,t)| \leq AC \exp\{B|x|^\beta\}, \qquad 0 \leq k \leq l.$$

is valid.

We note that what was said in Remarks 1, 2, 3 remains valid also with respect to Theorem 6.

The proofs of all the theorems listed use the conditions of uniqueness of the solution and correct solvability of the boundary-value problem in the infinite strip [1,2], as well as the theory of the spaces of G. E. Shilov (spaces of type S [3]).

Kharkov State University
named after A. M. Gorky

Received
24 IV 1968

CITED LITERATURE

[1] V. M. Borok, DAN, **183**, No. 5 (1968).
[2] V. M. Borok, DAN, **183**, No. 6 (1968).
[3] I. M. Gel' fand, G. E. Shilov, *Generalized Functions*, Vol. 2. *Spaces of Basic and Generalized Functions*, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.