ON AN INVERSE PROBLEM IN DIFFRACTION THEORY

MECHANICS

1969

SovietRxiv

Abstract

Full Text

UDC 534.88+517.946.9

MECHANICS

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ON AN INVERSE PROBLEM IN DIFFRACTION THEORY

(Presented by Academician L. I. Sedov on 24 VI 1968)

The scalar problem of diffraction of a plane wave $e^{ikr\cos\gamma}$ ($\cos\gamma = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos(\varphi-\beta)$; r, θ, φ are spherical coordinates; α, β are angles determining the direction of the incident wave) by a sphere of radius r_0 with center at the point $r = 0$ and with a variable admittance g, prescribed on the surface of the sphere, of the form

$$g(\theta, \rho) = \sum_{l=0}^{N-1} g_l(\rho) P_l(\cos\theta) \qquad (\operatorname{Re} g \geqslant 0) \tag{1}$$

(N is a given positive integer; $P_l(\cos\theta)$ are Legendre polynomials; $g_l(\rho)$ are rational functions of the parameter $\rho = kr_0$, which will subsequently be chosen in a special way) is formulated as follows: in the region $r > r_0$ outside the sphere it is required to find a function $p(r, \theta, \varphi, \alpha, \beta, N, k)$, meromorphically dependent on the parameter k, satisfying, with respect to the variables r, θ, φ, the Helmholtz equation

$$\Delta p + k^2 p = 0 \qquad (p \sim e^{-i\omega t}) \tag{2}$$

and the absorption condition [1–3]

$$\left|p - e^{ikr\cos\gamma}\right| < \infty \quad \text{for} \quad \operatorname{Im} k > |\operatorname{Re} k|, \tag{3}$$

and, on the surface of the sphere, the boundary condition

$$\partial p/\partial r + ikgp = 0 \quad \text{for } r = r_0. \tag{4}$$

For given coefficients $g_l(\rho)$ and number N, the formulated direct diffraction problem (1), (2), (3), (4) has a unique [2,3] solution $p(r, \theta, \varphi, \alpha, \beta, N, k)$.

Denote by $p_N(\rho, \alpha)$ the value of the function $p(r, \theta, \varphi, \alpha, \beta, N, k)$ at the point $r = r_0$, $\theta = 0$ of the surface of the sphere:

$$p_N(\rho, \alpha) = p(r_0, 0, \varphi, \alpha, \beta, N, k). \tag{5}$$

Owing to the axial symmetry of the domain and of the boundary condition, it does not depend on the angles φ, β.

In the present note a formulation is proposed of the inverse diffraction problem, consisting in the determination of such coefficients $g_l(\rho)$ entering into formula (1) that, as $\rho \to 0$, the ratio

$$f(\alpha, \rho) = p_N(\rho, \alpha)/p_N(\rho, 0)$$

approximates as accurately as possible a certain optimal function $f_N(\alpha)$.

As such a function we take

$$f_N(\alpha) = (N+1)^{-2} \sum_{n=0}^{N} (2n+1) P_n(\cos \alpha), \tag{6}$$

having, at the single point of the closed interval $0 \le \alpha \le \pi$, an absolute maximum (in modulus) $f_N(0) = 1$. The coefficients in the right-hand side of (6) are chosen so that, for the given natural number N and under the condition of uniqueness of the absolute maximum $f_N(0) = 1$, the functional of Yu. M. Sukharevskii ([4])

$$K_N = 2 \int_0^{\pi} |f_N(\alpha)|^2 \sin \alpha \, d\alpha \tag{7}$$

takes, for the function (6), the greatest possible value (in comparison with the values of this functional for functions of the form (6) with other real or complex coefficients)

$$K_N = (N+1)^2. \tag{8}$$

The condition that the absolute maximum of $f_N(\alpha)$ be attained at the single point $\alpha = 0$ is essential here. If two points are allowed, for example $f_N(0) = f_N(\pi) = 1$, then instead of (6), (8) we have

$$f_N(\alpha) = (N+1)^{-1}(2N+1)^{-1} \sum_{n=0}^{N} (4n+1) P_{2n}(\cos \alpha), \tag{6*}$$

$$K_N = (N+1)(2N+1). \tag{8*}$$

Here we shall confine ourselves to an example of the exact solution of such an optimal inverse problem for the simplest case $N = 1$, when, according to (1),

$$g(\theta, \rho) = g_0(\rho). \tag{9}$$

The solution of the direct diffraction problem (2), (3), (4), (9) has the form

$$p(r,\theta,\varphi,\alpha,\beta,1,k) = \sum_{n=0}^{\infty} i^n (2n+1) \left[j_n(kr) - \frac{j_n'(\rho) + ig_0 j_n(\rho)}{h_n'(\rho) + ig_0 h_n(\rho)} h_n(kr) \right] \times \\ \times \left[P_n(\cos\theta) P_n(\cos\alpha) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\alpha) \cos m(\varphi - \beta) \right],$$

where $j_n(\rho) = \sqrt{\pi/2\rho}\, J_{n+1/2}(\rho)$; $h_n(\rho) = \sqrt{\pi/2\rho}\, H^{(1)}_{n+1/2}(\rho)$ are spherical Bessel functions. In particular, according to (5), for the point $r = r_0$, $\theta = 0$ on the surface of the sphere we obtain

$$p_1(\rho,\alpha) = \sum_{n=0}^{\infty} i^n (2n+1) P_n(\cos\alpha) \left[j_n(\rho) - \frac{j_n'(\rho) + ig_0 j_n(\rho)}{h_n'(\rho) + ig_0 h_n(\rho)} h_n(\rho) \right] \\ = \sum_{n=0}^{\infty} \frac{i^{n+1}(2n+1) P_n(\cos\alpha)}{\rho^2 \left[h_n'(\rho) + ig_0 h_n(\rho) \right]}.$$

Hence, using the representation

$$h_n(\rho) = \frac{e^{i\rho}}{i^{n+1}\rho} \sum_{m=0}^{n} \frac{(-)^m}{m!} \frac{(n+m)!}{(n-m)!} \frac{1}{(2i\rho)^m},$$

we find the relation

$$f(\alpha,\rho) = \frac{p_1(\rho,\alpha)}{p_1(\rho,0)} = \frac{\sum_{n=0}^{\infty} (2n+1) c_n(\rho) (2i\rho)^{n}_{n}{}^{P}(\cos\alpha)}{\sum_{n=0}^{\infty} (2n+1) c_n(\rho) (2i\rho)^n}, \tag{10}$$

where it is denoted

$$\frac{1}{c_n(\rho)} = \sum_{m=0}^{n} \frac{(-)^m}{m!} \frac{(2n-m)!}{(n-m)!} (n - m + 1 - i\rho g_0 - i\rho)(2i\rho)^m. \tag{11}$$

If the admittance g_0 were a fixed complex number, then from (10), (11) it would follow that

$$f(\alpha,\rho) = 1 + O(\rho) \quad \text{as } \rho \to 0 \quad (P_0(\cos\alpha) \equiv 1).$$

However, if one chooses the coefficient g_0 in the form of a function of the parameter ρ, which in the present case is the solution of the inverse problem posed,

$$g_0(\rho) = 1 - i \cdot 2/\rho, \tag{12}$$

then from (10), (11) we obtain:

$$f(\alpha,\rho) = f_1(\alpha) + O(\rho) \quad \text{as } \rho \to 0, \tag{13}$$

which in the limit $\rho \to 0$ coincides with the optimal expression

$$f_1(\alpha) = \frac{1}{4}\sum_{n=0}^{1}(2n+1)P_n(\cos\alpha) = \frac{1+3\cos\alpha}{4},$$

given by formula (6) for $N = 1$, so that, according to (8), the maximum possible value (under an absolute maximum, at the single point $\alpha = 0$) of Sukharevsky's functional (7) is attained:

$$K_1 = 4. \tag{14}$$

If, however, an absolute maximum is allowed at two points $f_1(0) = f_1(\pi) = 1$, the solution of the inverse problem

$$g_0 = -i(3/\rho - \rho) \tag{15}$$

leads, according to (10), (11), (6), (8), to the function optimal for $N = 1$,

$$f(\alpha, 0) = f_1(\alpha) = (5\cos^2\alpha - 1)/4, \tag{16}$$

which gives the maximum value of the functional (7)

$$K_1 = 6. \tag{17}$$

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Received
19 VI 1968

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Note: Figure translations are in progress. See original paper for figures.