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Abstract

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A PRIORI ESTIMATES OF SOLUTIONS OF ELLIPTIC EQUATIONS IN THE CLASS OF ANALYTIC FUNCTIONS AND THEIR APPLICATIONS TO THE CAUCHY-POISSON PROBLEM

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In this paper estimates are established, up to the boundary, for solutions of elliptic equations in the class of functions analytic in the tangential directions. As an application, the Cauchy-Poisson problem on the motion of a fluid with a free surface is considered in a rigorous formulation. After the estimates have been established in the class of analytic functions, the existence and uniqueness theorem for the solution of the Cauchy-Poisson problem follows almost immediately from the works of L. V. Ovsyannikov (^{1,2}), Leray and Ohya (^{3}), which generalize the Cauchy-Kovalevskaya theorem.

1. Norms and their properties. We shall use the notation
$\beta = (\beta_0, \beta_1, \dots, \beta_n)$, $|\beta| = \beta_0 + \beta_1 + \dots + \beta_n$, $x = (x_0, x_1, \dots, x_n)$,
$D^\beta = \partial^{|\beta|}/\partial x_0^{\beta_0} \cdots \partial x_n^{\beta_n}$.

Let $y = (y_1, \dots, y_m)$; let $Y \subset R^m$ be an open set; let $\Omega \subset R^{n+1}$ be either an open set, or a hyperplane, or the union of an open set and several hyperplanes. For functions $f(x) : \Omega \to R$ and $F(x, y) : \Omega \times Y \to R$ define

$$\|f, \Omega\|_{\rho,k+\alpha} = \sum_{l=0}^{\infty} \frac{\rho^l}{l!} \max_{|\beta|=l,\ \beta_0=0} \|D^\beta f\|_{C_{k+\alpha}(\Omega)},$$

$$\|F, \Omega \times Y, \nu, \theta\|_{\rho,k} = C(1 + \nu_1 + \dots + \nu_m)^{k+1} \sum_{l,\sigma} \frac{\rho^l \theta^\sigma}{l!\sigma!} \max_{|\beta|=l,\ \beta_0=0} \times$$

$$\times \|D_x^\beta D_y^\sigma F\|_{C_{k+1}(\Omega \times Y)}.$$

Here $\nu = (\nu_1, \dots, \nu_m)$, $\theta = (\theta_1, \dots, \theta_m)$, $\theta^\sigma = \theta_1^{\sigma_1} \cdots \theta_m^{\sigma_m}$, $\sigma! = \sigma_1! \cdots \sigma_m!$. The constant $C = C(k)$. The norm of a differential operator is defined as the sum of the norms of its coefficients.

Let us formulate the basic properties of the introduced norms.
If $f, g : \Omega \to R$, then

$$\|D_j f, \Omega\|_{\rho,k+\alpha} \ll \frac{\partial}{\partial \rho} \|f, \Omega\|_{\rho,k+\alpha}, \qquad 0 < j \leq n, \tag{1}$$

$$\|fg, \Omega\|_{\rho,k+\alpha} \ll \|f, \Omega\|_{\rho,k+\alpha} \|g, \Omega\|_{\rho,k+\alpha}. \tag{2}$$

For a differential operator $a(x, D)$ define

$$\|[a]f, \Omega\|_{\rho,k,\alpha} = \sum_{l=1}^{\infty} \frac{\rho^l}{l!} \max_{|\beta|=l,\ \beta_0=0} \|([aD^\beta - D^\beta a]f)\|_{C_{k+\alpha}(\Omega)}.$$

If the dimension of Ω is equal to $n+1$, then

$$\|[a]f, \Omega\|_{\rho,k+\alpha} \ll [\|a, \Omega\|_{\rho,k+\alpha} - \|a, \Omega\|_{0,k+\alpha}] \|f, \Omega\|_{\rho,k+m+\alpha}; \tag{3}$$

m is the order of the operator $a(x, D)$.

Let $F(x, y) : \Omega \times Y \to R$, $V(x) : \Omega \to Y$. Then

$$\|F \circ V, \Omega\|_{\rho,k+\alpha} \ll \|F, \Omega \times Y, \|V, \Omega\|_{0,k+\alpha}, \|V, \Omega\|_{\rho,k+\alpha} - \|V, \Omega\|_{0,k+\alpha}\|_{\rho,k}, \tag{4}$$

where $F \circ V = F(x, V(x))$, and $\|V, \Omega\|_{\rho,k+\alpha}$ is the vector with coordinates $\|V_j, \Omega\|_{\rho,k+\alpha}$.

Inequalities (1)-(3) follow from the definitions and Leibniz' formula. The proof of (4) is omitted for lack of space.

In what follows we shall say: the series

$$F(t, \tau, \rho, \theta) = \sum_{s,\sigma} \frac{\rho^s \theta^\sigma}{s! \sigma!} F_{s\sigma}(t, \tau)$$

as a function of $t, \rho, \tau = (\tau_1, \dots, \tau_N)$, $\theta = (\theta_1, \dots, \theta_N)$ belongs to the space Γ, if it converges and is continuous in t, τ in a neighborhood of the point $t = \rho = 0$, $\tau = \theta = 0$.

If $\Phi(t, \rho) = (\Phi_1, \dots, \Phi_N)$, then by $F(\Phi)$ we shall denote the function $F(t, \Phi(t, 0), \Phi(t, \rho) - \Phi(t, 0))$. It is clear that $F(\Phi) \in \Gamma$, if $F, \Phi \in \Gamma$.

The function $f(t,x) : \{0,T\} \times \Omega \to R$ belongs to the space $B^p_{k+\alpha}(\Omega)$, if $\|D^j_t f; \Omega\|_{\rho,k+\alpha} \in \Gamma$ for $j \leq p$. The operator $L(t,x,D) \in B^p_{k+\alpha}(\Omega)$, if its coefficients $a_\beta(t,x) \in B^p_{k+\alpha}(\Omega)$. Similarly, $F(t,x,y) \in B^p_k(\Omega, Y)$, if $\|D^{jF}_t, \Omega \times Y, \tau, \theta\|_{\rho,k} \in \Gamma$ for $j \leq p$. $L(t,x,y,D) \in B^p_k(\Omega, Y)$, if $a_\beta(t,x,y) \in B^p_k(\Omega, Y)$.

From inequalities (1), (2), and (4) we have:

$$D^\beta_x D^q_t : B^p_{k+\alpha}(\Omega) \to B^{p-q}_{k-\beta_0+\alpha}(\Omega), \qquad q \leq p, \quad \beta_0 \leq k.$$

$$f \cdot g \in B^p_{k+\alpha}(\Omega), \quad \text{if } f, g \in B^p_{k+\alpha}(\Omega).$$

$$F \circ V \in B^p_{k+\alpha}(\Omega), \quad \text{if } f \in B^p_{k+\alpha}(\Omega), \quad F \in B^p_k(\Omega \times Y),$$

$$f(t,x) : \{0,T\} \times \Omega \to Y.$$

2. Estimates of solutions of elliptic equations. Introduce the notation: $X = \{x = (x_0, x_1, \dots, x_n) : 0 \leq x_0 \leq 1,\ -\infty < x_1, \dots, x_n < \infty\}$; $S_j = \{x : x_0 = j,\ -\infty < x_1, \dots, x_n < \infty\}$, $j = 0, 1$; $\overline{X} = X \cup S_0 \cup S_1$.

In the layer X the problem to be considered is

$$\begin{aligned} L(x,D)u &= \sum_{|\beta| \leq m} a_\beta(x) D^\beta u = f, && x \in X; \\ B_0(x,D)u &= \sum_{|\beta| \leq l_0} a^0_\beta(x) D^\beta u = \varphi_0, && x \in S_0; \\ B_1(x,D)u &= \sum_{|\beta| \leq l_1} a^1_\beta(x) D^\beta u = \varphi_1, && x \in S_1. \end{aligned} \tag{5}$$

It is assumed that the coefficients of the operator L and B_j are infinitely differentiable and bounded, L is uniformly elliptic, and B_j satisfy the complementing condition (see (4)).

Theorem 1. *Let the conditions listed above be fulfilled. Then, for* $k \geq \max(m, l_0, l_1)$, *the inequality*

$$\|u, \overline{X}\|_{\rho,k+\alpha} \leq A(\rho) \left[\|Lu, \overline{X}\|_{\rho,k-m+\alpha} + \sum_{j=0}^{1} \|B_j u, S_j\|_{\rho,k-l_j+\alpha} + \|u\|_{C_0(X)} \right], \tag{6}$$

holds, where

$$A(\rho) = C\left\{1 - C\left[\rho + \|L, \overline{X}\|_{\rho,k-m+\alpha} - \|L, \overline{X}\|_{0,k-m+\alpha} + \sum_{j=0}^{1}(\|B_j, S_j\|_{\rho,k-l_j+\alpha} - \|B_j, S_j\|_{0,k-l_j+\alpha})\right]\right\}^{-1}$$

and the constant C depends on n, α; the ellipticity constant of the operator L; the constants characterizing fulfillment of the complementing condition, and the norms of the coefficients of the operators L and B_j in the spaces $C_{k-m+\alpha}(\overline{X})$ and $C_{k-l_j+\alpha}(S_j)$, respectively.

The proof is carried out in the same way as in (5) when deriving estimates

solutions of the Dirichlet problem. It is based on a Schauder-type estimate

$$\|V\|_{C_{k+\alpha}(\bar{X})} \leq C\left[\|LV\|_{C_{k-m+\alpha}(\bar{X})} + \sum_{j=0}^{1}\|B_{jV}\|_{C_{k-l_j+\alpha}(S_j)} + \|V\|_{C_0(\bar{X})}\right], \qquad (7)$$

which follows from the results of [4]. Putting $V = D^{\beta}u,\ \beta_0 = 0$, we obtain (6) from (3) and (7).

Corollary. If $m = 2,\ B_0 = D_0 + \sum_{j=1}^{n} a_j^0 D_j,\ a_{0\ldots0} \leq 0$ and $B_1 \equiv 1$, then the term $\|u\|_{C_0(\bar{X})}$ in the right-hand side of inequality (6) may be omitted.

The required assertion is given by the following

Lemma 1. Let the operator L be uniformly elliptic; $a_{0\ldots0} \leq 0$; $m = 2$, the coefficients of the operators L and

$$B_0 = D_0 + \sum_{j=1}^{n} a_j^0(x) D_j$$

be bounded and continuous, and let $B_1 \equiv 1$. If $u \in C_2(\bar{X})$, then

$$|u(x)| \leq C\left[\|Lu\|_{C_0(\bar{X})} + \|B_0 u\|_{C_0(S_0)} + \|u\|_{C_0(S_1)}\right],$$

where the constant C depends on the ellipticity constant of the operator L and on an exact upper bound for the moduli of the coefficients of the operators L and B_0.

Proof. Introduce

$$g(x) = \sum_{j=1}^{n} \ln\left[(x_0+1)^2 + x_j^2\right]; \qquad \lambda = \sup_{x\in\bar{X}}(|(L - a_{0\ldots0})g|,\ |B_0 g|);$$

$$u_\varepsilon(x) = u(x) + (H + \lambda\varepsilon)e^{\alpha x_0} - \varepsilon g(x); \qquad H = \|Lu\|_{C_0(\bar{X})} + \|B_0 u\|_{C_0(S_0)}.$$

Choose $\alpha > 1$ so that $e^{-\alpha x_0} L e^{\alpha x_0} > 1$. Then $Lu_\varepsilon > 0$, $Bu_\varepsilon > 0$, and, by the boundedness of u, $u_\varepsilon < 0$ for sufficiently large $R^2 = x_1^2 + \dots + x_n^2$. The required result follows from the well-known maximum principle for bounded domains.

3. Waves on the surface of a liquid. Consider potential motion in a field of external forces in the domain

$$\Omega_t = \{x = (x_0, x_1, x_2) : \ h(x_1, x_2) < x_0 < \zeta(t, x_1, x_2), \ -\infty < x_1, x_2 < \infty\},$$

where $x_0 = \zeta(t, x_1, x_2)$ is the free surface, and $x_0 = h(x_1, x_2)$ is the bottom.

It is known that ζ and the potential u satisfy the system of equations

$$\begin{aligned}
\Delta u &= 0, \qquad x \in \Omega_t;\\
du/dN &= 0, \qquad x_0 = h(x_1, x_2);\\
\partial u/\partial t &= -\tfrac{1}{2}|\nabla u|^2 + F(t, x), \qquad x_0 = \zeta(t, x_1, x_2);\\
\partial \zeta/\partial t &= -\nabla u \cdot \nabla \zeta + \partial u/\partial x_0, \qquad x_0 = \zeta(t, x_1, x_2).\\
\zeta(0, x_1, x_2) &= \zeta_0(x_1, x_2); \quad u(0, \zeta_0(x_1, x_2), x_1, x_2) = u_0(x_1, x_2).
\end{aligned}$$

Here N is the normal to the surface $x_0 = h(x_1, x_2)$.

Theorem 2. Let $h, \zeta_0, u_0 \in B^0_{2+\alpha}(\mathbb{R}^2)$, $F(t, \zeta_0(x_1, x_2) + y, x_1, x_2) \in B^0_2(X, \{|y| < \delta\})$, and $0 < \delta_0 \le \zeta_0 - h \le d < \infty$. Then, for sufficiently small times t, the system written above has a unique solution (ζ, u), continuously differentiable in t and analytic in x.

Proof. Make the change of variables

$$x' = (x'_0, x'_1, x'_2); \qquad x'_0 = \frac{x_0 - h(x_1, x_2)}{\zeta(t, x_1, x_2) - h(x_1, x_2)};$$

$$x'_1 = x_1; \qquad x'_2 = x_2;$$

$$\zeta'(t, x'_1, x'_2) = \zeta(t, x'_1, x'_2) - \zeta_0(x'_1, x'_2);$$

$$u'(t, x') = u(t, h + (\zeta - h)x'_0, x'_1, x'_2) - u'_0(h + (\zeta - h)x'_0, x'_1, x'_2).$$

Here $u'_0(x_0, x_1, x_2)$ is the solution of the problem

$$\begin{aligned}\Delta u_0' &= 0, && x \in \Omega_0;\\ du_0'/dN &= 0, && x_0 = h(x_1, x_2);\\ u_0' &= u_0, && x_0 = \zeta(x_1, x_2).\end{aligned}$$

Rewrite the original system, omitting primes and denoting any function f, depending on t, x, φ and on derivatives of φ of order not higher than q, by $f(D^q\varphi)$:

$$L(D^2\zeta, D)u = \sum_{0<|\beta|\le 2} a_\beta(D^2\zeta)D^\beta u = L(D^2\zeta, D)u_0, \qquad x \in X;$$

$$B(D\xi, D)u = \left[D_0 + \sum_{j=1}^{2} a_j(D\xi)D_j\right] u = B(D\xi, D)u_0, \qquad x \in S_0;$$

$$\begin{aligned}&\partial\xi/\partial t = a(D\xi, Du), && x \in S_1;\\ &\partial u/\partial t = b(D\xi, Du), && x \in S_1;\\ &u(0, x) = 0, && x \in X;\\ &\xi(0, x) = 0, && x \in S_1,\end{aligned}$$

where

$$L(0, D)u_0 = 0,\ x \in X; \qquad B(0, D)u_0 = 0,\ x \in S_0; \qquad u_0 = u_0(x_1, x_2),\ x \in S_1.$$

From the substitution it is clear that

$$\begin{aligned}L(y, D) &\in B^0(\bar{X}, Y), && B(y, D) \in B_1^0(S_0, Y),\\ a(y) &\in B_2^0(S_1, Y), && b(y) \in B_2^0(S_1, Y),\end{aligned}$$

where Y is a ball with center at the origin whose radius is less than δ_0.

According to Theorem 1, $u_0 \in B_{2+\alpha}^0(\bar{X})$.

In order to find u, it is sufficient to know ξ and $v = u|_{S_1}$. Let $\xi, v \in B_{2+\alpha}^0(S_1)$; $\|\xi, S_1\|_{\rho,2+\alpha} = \psi(t, \rho)$, $\psi(0, \rho) = 0$; $\|v, S_1\|_{\rho,2+\alpha} = \Phi(t, \rho)$, $\Phi(0, \rho) = 0$.

Introduce

$$P(\xi, v) = \partial u/\partial x_0|_{S_1},$$

where u is the solution of the problem

$$\begin{aligned} L(0,D)u &= [L(0,D)-L(D^2\xi,D)]u + L(D^2\xi,D)u_0, && x\in X;\\ B(0,D)u &= [B(0,D)-B(D\xi,D)]u + B(D\xi,D)u_0, && x\in S_0;\\ u &= v, && x\in S_1. \end{aligned}$$

According to Lemma 1, for sufficiently small t there exists a unique $u \in C_{2+\alpha}(\bar{X})$. Applying inequality (4),

$$\|L(0,D)-L(D^2\xi,D),\bar{X}\|_{\rho,\alpha} \ll A_1(\psi)\psi \in \Gamma;$$

$$\|B(0,D)-B(D\xi,D),S_0\|_{\rho,1+\alpha} \ll A_2(\psi)\psi \in \Gamma;$$

$$\|B(D\xi,D)u_0,S_0\|_{\rho,1+\alpha} \ll A_3(\psi)\psi \in \Gamma; \qquad \|L(D^2\xi,D)u_0,\bar{X}\|_{\rho,\alpha} \ll A_4(\psi)\psi \in \Gamma,$$

from the a priori estimate (6) and the corollary to Theorem 1 we obtain:

$$\|u,\bar{X}\|_{\rho,2+\alpha} \ll F_1(\psi,\Phi) \in \Gamma, \qquad F_1(0,0)=0,$$

i.e.

$$u \in B^0_{2+\alpha}(\bar{X}); \qquad u(0,x)=0,\ x\in X.$$

Since $\|f,S_1\|_{\rho,k+\alpha} \ll \text{const}\,(1+\partial/\partial\rho)\|f,S_1\|_{\rho,k-1+\alpha}$, it follows that

$$\|P(\xi,v),S_1\|_{\rho,2+\alpha} \ll (1+\partial/\partial\rho)F(\psi,\Phi); \qquad F\in\Gamma, \quad F(0,0)=0.$$

Thus the original problem is reduced to finding (ξ, v) from the system

$$\begin{aligned} \partial\xi/\partial t &= a(D\xi,Dv,P(\xi,v)), && x\in R^2;\\ \partial v/\partial t &= b(D\xi,Dv,P(\xi,v)), && x\in R^2;\\ \xi(0,x) &= v(0,x)=0, && x\in R^2, \end{aligned}$$

where $a \in B^0_2(R^2_2, Y)$, $b \in B^0_2(R^2_2, Y)$, and for the operator $P(\xi, v)$ the estimate

$$\|P(\xi,v),R^2\|_{\rho,2+\alpha} \ll (1+\partial/\partial\rho)F(\|\xi,R^2\|_{\rho,2+\alpha},\ \|v,R^2\|_{\rho,2+\alpha}), \qquad F(0,0)=0$$

is valid.

The theorem on the existence and uniqueness of the solution (ξ, v) follows almost immediately from the results of works (1–3). It can be proved, like the Cauchy-Kovalevskaya theorem, by successive approximations.

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Note: Figure translations are in progress. See original paper for figures.

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