# REPRESENTING SPACES FOR THE \(K\)-FUNCTOR WITH COEFFICIENTS

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## Abstract

## Full Text

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*MATHEMATICS*

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# REPRESENTING SPACES FOR THE $K$-FUNCTOR WITH COEFFICIENTS

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This paper describes the connection between the Adams–Toda homotopy operations (see [1], Theorem 1.7) and the differentials in the Atiyah–Hirzebruch spectral sequence (see [2]). The results obtained make it possible to describe the homotopy type of representing spaces for $K$-theory with coefficients in the rings $Z_p$ and $Z[t]/(t+1)^p - 1$, where $p$ is an odd prime.

We shall consider only complex $K$-theory. The Atiyah–Hirzebruch spectral sequence for a complex $X$ will be denoted by $SP(X)$ for the $K$-functor and by $SP(X; Z_p)$ for the $K$-functor mod $p$. We note that by a cellular complex we shall mean a cellular complex with finite skeleta.

I. **Lemma 1.** *For any $n > 2p-3$ there exists a sequence of mappings*

$$A(n) = \{\alpha_s; \alpha_s : S_p^{n+2s(p-1)-1} \to S_p^n\}$$

*such that: a)*

$$\alpha_s = (\Sigma^{2s(p-1)}\alpha_1) \circ \cdots \circ \alpha_1;$$

*b) for the complex*

$$W_s = S_p^n \cup_{\alpha_s} D_p^{n+2s(p-1)},$$

*in the spectral sequences $SP(W_s)$ and $SP(W_s, Z_p)$ the differential $d_{2s(p-1)+1}$ acts nontrivially. Here*

$$S_p^n = S^n \cup_p D^{n+1}.$$

**Proof.** Consider a mapping $f : S_p^{2p} \to S^3$ with Hopf invariant 1 mod $p$ (see [3]). It is easy to show that for $n > 2p-3$ the mapping $\Sigma^{n-3} f$ decomposes into the composition

$$\pi \circ \alpha_1 : S_p^{n+2p-3} \to S_p^{n-1}.$$

Since

$$d_{2p-1} = e_{2,1},$$

we have $K^*(W_1) = 0$, and the mapping

$$\alpha_1^* : K^*(S_p^{n-1}) \to K^*(S_p^{n+2p-3})$$

is an isomorphism. Consequently, for every $s$ the mapping $\alpha_s^*$ is an isomorphism and $K^*(W_s) = 0$. The lemma is proved.

**Theorem 1.** *For any $n > 2p-3$ there exists a cellular complex $Y(Z_p, n)$ such that:*

a)
$$\pi_q(Y(Z_p, n)) = \begin{cases} Z_p, & q \equiv n \pmod{2(p-1)}, \\ 0, & q \not\equiv n \pmod{2(p-1)}; \end{cases}$$

b) *the generating element*
$$u_n \in H^n(Y(Z_p, n), Z_p)$$
*is a cycle of all differentials in $SP(Y(Z_p, n); Z_p)$.*

*The complex $Y(Z_p, n)$ has the following properties:*

1) *All cellular complexes $Y$ satisfying conditions a) and b) are homotopy equivalent to the complex $Y(Z_p, n)$.*

2) *A mapping*
$$f : Y(Z_p, n) \to Y(Z_p, n)$$
*is a homotopy equivalence if and only if $f^*(u_n) \neq 0$.*

**Proof.** From the sequence $A(n) = \{\alpha_s\}$ one can construct a sequence of cellular complexes $\{Y_s\}$ as follows. As $Y_0$ take the Eilenberg-Mac Lane space $K(Z_p, n)$, $n > 2p-3$; as $Y_1$, take the space of the fibration
$$Y_1 \xrightarrow{k(Z_p, n+2(p-1))} Y_0$$
with Postnikov invariant $e_{2,1}$. Denote by $a_1$ the generator of the group $H^n(Y_1; Z_p)$, and by $b_1$ the generator of the group $\pi_n^p(Y_1)$. By construction, in $SP(Y_1; Z_p)$ the element
$$a_1 \in E_{2p-1}^{n,2t}$$
is a cycle of the differential $d_{2p-1}$, and therefore the generator of the group $\pi_{n+2(p-1)}(Y_1)$ can be decomposed into the composition
$$S^{n+2(p-1)} \subset S_p^{n+2(p-1)} \xrightarrow{\alpha_1} S_p^n \subset Y_1.$$
According to Theorem 2 of ([5]), the element
$$a_1 \in E_{2p-1}^{n,2t}$$
is a cycle of the differentials $d_r$ for $r < 4(p-1)+1$

The embedding $b_1 : S^n_p \subset Y_1$ extends to a map $\tilde{b}_1 : W_2 \to Y_1$, and since, by Lemma 1, the differential $d_{4(p-1)+1}$ acts nontrivially in the complex $W_2$, we obtain that $d_{4(p-1)+1}(a_1) \neq 0$. Thus, in the group

$$E_{4(p-1)+1}^{n-4(p-1)+1,\, 2t-4(p-1)} \subset SP(Y_1, Z_p)$$

an element is marked. Take as $Y_2$ the fibration space

$$Y_2 \xrightarrow{K(Z_p,\, n+4(p-1))} Y_1$$

with Postnikov invariant $c_2 \in H^{n+4(p-1)+1}(Y_1, Z_p)$, where $c_2$ is some representative of the element $d_{4(p-1)+1}(a_1)$ in cohomology. Denote by $a_2 \in H^n(Y_2, Z_p)$ and $b_2 \in \pi^p_n(Y_2)$ the generating elements. Using the complex $W_2$, we obtain that the generating group $\pi_{n+4(p-1)}(Y_2) = Z_p$ decomposes into the composition

$$S^{n+4(p-1)} \subset S_p^{n+4(p-1)} \xrightarrow[b_2]{a_2} S^n_p \subset Y_2.$$

Continuing the construction, we obtain a sequence of complexes

$$\{\, Y_i,\ Y_i \xrightarrow{K(Z_p,\, n+2i(p-1))} Y_{i-1} \,\}, \quad i \geq 0.$$

The $(n + 2i(p-1) - 1)$-dimensional skeleta of the spaces $Y_i$ and $Y_{i-1}$ can be identified for any $i$, and therefore we have a sequence of embedded cell complexes

$$[Y_0]_{n+2(p-1)-1} \subset [Y_1]_{n+4(p-1)-1} \subset \cdots \subset [Y_i]_{n+2(i+1)(p-1)-1} \subset \cdots.$$

Put

$$Y(Z_p, n) = \varinjlim [Y_i]_{n+2(i+1)(p-1)-1}.$$

It is clear that $Y(Z_p, n)$ is a cell complex. Denote by $u_n \in H^n(Y(Z_p, n), Z_p)$ a generator of the group.

By construction, $u_n$ is a cycle of all differentials in $SP(\ , Z_p)$, and moreover, for any cell complex $X$ and element $x \in H^n(X, Z_p)$ that is a cycle of all differentials in $SP(\ , Z_p)$, there exists a continuous map $f : X \to Y(Z_p, n)$ such that $f^* u_n = x$. Using now the fact that, for any $s$, the generator of the group $\pi_{n+2s(p-1)}(Y(Z_p, n))$ decomposes into the composition

$$S^{n+2s(p-1)} \to S_p^{n+2s(p-1)} \xrightarrow{\alpha_s} S^n_p \subset Y(Z_p, n),$$

we obtain the proof of properties 1) and 2) of the space $Y(Z_p, n)$. The theorem is proved.

Since the spectral sequence $SP(Y, (Z_p, n)Z_p)$ converges strongly ([4]), in the group $k^n(Y(Z_p, n)Z_p)$ there exists an element $\zeta_n$ of filtration $n$.

**Lemma 2.** a) For any $n > 2p-3$ there exists a homotopy equivalence

$$\chi_n : \Omega Y(Z_p, n+1) \xrightarrow{\simeq} Y(Z_p, n).$$

b) There exists a homotopy equivalence

$$\chi : \Omega^{2(p-1)} Y(Z_p, 2(p-1)) \to Z_p \times Y(Z_p, 2(p-1)).$$

**Proof.** Let $K_p$ be a representing space for the $K$-functor mod$p$. Denote by $K_p(2n)$ the $2n$-decreasing space for the complex $K_p$, and by $K_p(2n-1)$ the space $\Omega K_p(2n)$. According to the classifying properties of the spaces $K_p(n)$, there exists a map $f : Y(Z_p, n+1) \to K_p(n+1)$ such that $f^*(\eta_{n+1}) = \zeta_{n+1}$, where $\eta_{n+1}$ is the universal element. Consider the map

$$\Omega f : \Omega Y(Z_p, n) \to \Omega K_p(n).$$

Since the homomorphism

$$(\Omega f)^* : H^n(\Omega K_p(n+1), Z_p) \to H^n(\Omega Y(Z_p, n+1), Z_p)$$

is nonzero, the generating element

$$c \in H^n(\Omega Y(Z_p, n+1), Z_p)$$

is, obviously, a cycle of all differentials in $SP(\ , Z_p)$. Applying now Theorem 1, we obtain that the spaces $\Omega Y(Z_p, n+1)$ and $Y(Z_p, n)$ are homotopy equivalent. Part a) is proved. Part b) is proved analogously.

**Corollary 1.** For any $n > 2p-3$ there exists a homotopy equivalence $\lambda$:

$$\prod_{k=0}^{p-1} Y(Z_p, n+2k) \to K_p(n).$$

Put

$$h_p^{-q}(X, Y) = [S^q(X/Y),\ Z_p \times Y(Z_p, 2(p-1))]_*, \quad q > 0.$$

The homotopy equivalence

$$\chi : \Omega^{2(p-1)} Y(Z_p, 2(p-1)) \to Z_p \times Y(Z_p, 2(p-1))$$

makes it possible to define the groups $h_p^{-q}$ also for negative-

$q$. By the standard method it is easy to verify that the homotopy functor $h_p^*$ defines an extraordinary cohomology theory $(2(p-1))$-periodic). Applying Lemma 2 and Corollary 1, we obtain the theorem:

**Theorem 2.** *On the category of cellular complexes there exists a* $2(p-1)$*-periodic cohomology theory* $h_p^* = \sum h_p^q$ *such that for any pair of cellular complexes* $(X, Y)$ *there is a natural isomorphism*

$$\tau_p : \sum_{i=0}^{p-1} h_p^{-2i}(X,Y) \xrightarrow{\approx} k^0(X,Y;\mathbb{Z}_p).$$

**II.** Let $L_p^\infty$ be the infinite-dimensional lens space. Denote by $L^n$ the $2n$-dimensional skeleton of the space $L_p^\infty$. Denote the group $k^0(L^n)$ by $A_n$ and put, as usual, $\tilde{k}^0(X;A_n) = k^0(X\#L^n)$. Since $L^1 = S_p^1$, it follows that $k^*(\,;A_1) = k^*(\,;\mathbb{Z}_p)$. For any $n$ the equality $L^{n+1}/L^n = S_p^{2n+1} = S^{2n}S_p^1$ holds, which makes it possible to generalize Theorem 2.

**Theorem 3.** *On the category of cellular complexes, for every $n > 0$ there exist $2(p-1)$-periodic cohomology theories $h_{p,n}^*$ and natural transformations $\tau_{p,n}^0 : h_{p,n}^0 \to k^0(\,;A_n)$, satisfying the following conditions:*

a) *for any pair of cellular complexes there is an isomorphism*

$$\tau_p : \sum_{i=0}^{p-1} h_{p,n}^{-2i}(X,Y) \to k^0(X,Y;A_n),$$

b) *there are transformations $p_n : h_{p,n} \to h_{p,n-1}$ such that $\pi_n^*\tau_{p,n} = \tau_{p,n-1}p_n$, where the projection $\pi_n : k^*(\,;A_n) \to k^*(\,;A_{n-1})$ is induced by the inclusion $\pi_n : L^{n-1} \to L^n$.*

**Proof.** Denote by $V^n$ the space of continuous mappings $\{L^n \to BU\}_*$ preserving the base point. Since $L^n$ is a finite complex, the space $V^n$ is a representing space for the $k^*(\,;A_n)$-theory. Suppose that for all $q \le n$ cellular complexes $Y_q$ and mappings $p_q : Y_q \to Y_{q-1}$, $\tau_q^0 : Y_q \to V^q$ have been constructed such that $\tau_{q-1}^0 p_q \simeq f_{q-1}\tau_q^0$ and $\tau_q^0$ induces an isomorphism of the groups $\pi_{2s(p-1)}$, $s \ge 0$. (For $n = 1$, $V^1 = K_p$, therefore as $Y_1$ one may take $Y(\mathbb{Z}_p, 2(p-1))$.) We construct the complex $Y_{n+1}$ and mappings $p_n$ and $\tau_{n+1}^0$ satisfying these conditions. Consider the segment of the Puppe sequence

$$(\pi_n : L^n \subset L^{n+1}) : \; S^{2n+1}S_p^1 \to S^2L^n \to S^2L^{n+1}.$$

There is a Serre fibration $g : \Omega^2V^n \to \Omega^{2n+1}K_p$ with fiber $F^a \simeq \Omega^2V^{n+1}$, and moreover $i = \Omega^2 f_n \cdot a$, where $i : F \to \Omega^2V^n$ is the inclusion of the fiber. Using standard homotopy technique, it is easy to show that $F$ is homotopy equivalent to the fibration space $F' \to \Omega^2V^n$, induced by the universal Serre fibration $* = p\Omega^{2n+1}K_p* \to \Omega^{2n+1}K_p$ by means of the mapping $g$. Consider the composition of mappings

$$\varphi : \; \Omega^2(Y_n)_0 \xrightarrow{\tau_n^0} \Omega^2(V^n)_0 \xrightarrow{g} \Omega^{2n+1}(K_p)_0 \xrightarrow{\Omega^{2n+1}} \Omega^{2n+1}Y(\mathbb{Z}_p, 2p-1)$$

and denote by $Y_{n+1}$ the fibration space $Y_{n+1} \to \Omega^2(Y_n)_0$, induced by the fibration $* \to \Omega^{2n+1}Y(\mathbb{Z}_p, 2p-1)$ by means of the mapping $\varphi$. It follows directly from the construction that there exist mappings $p_n$ and $\tau^0_{n+1}$ satisfying the required conditions. The induction step is complete.

Put $h^{-q}_{p,n}(X,Y) = [S^{qX}/Y, A_n \times Y]$. It is easy to verify that the cohomology theory $h^*_{p,n}$ and the transformation $(\tau^0_n)^* : h_{p,n} = k^0(\,;A_n)$ satisfy all the conditions of the theorem.

**III.** We now consider the $K$-functor with coefficients in the ring $k^*(L^\infty_p)$. Since $H^*(L^\infty_p; Q) = 0$, we have $k^*(L^\infty_p) = \mathcal{K}^*(L^\infty_p)$ (see (4)). It is easy to show that $\mathcal{K}^1(L^\infty_p) = 0$ and $\mathcal{K}^0(L^\infty_p) = \mathbb{Z}[t]/(t+1)^p - 1$, where $t = \zeta - 1$, and $\zeta$ is the canonical line bundle over $L^\infty_p$. Denote the ring

$\mathbb{Z}[t]/(t+1)^p - 1$ by $A$. The ring $A$ is additively isomorphic to the direct sum of $(p-1)$ copies of the ring of integral $p$-adic numbers, and therefore in questions where consideration of the $K$-functor with $p$-adic coefficients is required, one may use the functor $k(\,;A)$ (see ([4]), § 8). In ([4]) it is shown that if $X$ is a finite complex, then there is an isomorphism

$$k^*(X;A) \simeq K^*(X) \otimes_Z A.$$

For an arbitrary complex $X$ there exists an exact sequence relating the groups $k^*(X;A)$ and $k^*(X)$. From the results of Section II it follows:

**Theorem 4.** *On the category of cellular complexes there exists a $2(p-1)$-periodic cohomology theory $\mathcal{H}_p$ and a natural transformation $\tau^0_p : \mathcal{H}^0_p \to k^0(\,;A)$, which for any pair of complexes $(X,Y)$ induces an isomorphism*

$$\tau_p : \sum_{i=0}^{p-1} \mathcal{H}^{-2i}_p(Z,Y) \to k^0(X;A).$$

**Proof.** From Theorem 3 it follows that for any pair of cellular complexes $(X,Y)$ and any integer $q$ an inverse sequence of groups $\{h^q_{p,n}(X,Y)\}_{n>0}$ is defined. Since $H^*(L^n;Q) = 0$, for any cellular complex $X$ there is an isomorphism

$$k^q(X;A_n) \simeq \varprojlim k^q(X_l;A_n),$$

where $X_l$ is the $l$-dimensional skeleton of the complex $X$. For every $l$ the group $k^q(X_l;A_n)$ is finite; hence the group $k^q(X;A_n)$ is profinite. It follows that the group $h^q_{p,n}(X) \subset k^q(X;A_n)$ is also profinite. Put

$$\mathcal{H}^q_p(X,Y) = \varprojlim h^q_{p,n}(X,Y).$$

Using ([4]), § 3, we obtain that the functor $\mathcal{H}^*_p = \sum \mathcal{H}^q_p$ defines a cohomology theory on the category of cellular complexes. For any complex $X$ there is an isomorphism

$$k^*(X \# L^\infty_p) \simeq K^*(X \# L^\infty_p)$$

(see [4]). Clearly,

$$k^*(X\#L_p^\infty) \simeq \lim_{\leftarrow} k^*(X\#L^n).$$

Thus, the sequences of transformations $\{\tau_{p,n}^0\}$ and $\{\tau_{p,n}\}$ induce transformations $\tau_p^* : \mathcal{H}_p^0 \to k^0(\ ; A)$ and

$$\tau_p : \sum \mathcal{H}_p^{-2i}(X, Y) \to k^0(X; A).$$

The theorem is proved.

IV. In Section II we introduced a sequence of fibrations in the sense of Serre

$$\{f_n : V \to V^n,\ f_n^{-1}(*) \simeq \Omega^{2n} K_p\}.$$

Put $V = \lim_{\leftarrow} V^n$. Clearly, for every $n$ the projection $\pi_n : V \to V^n$ is a fibration in the sense of Serre. All $V^n$ are $H$-spaces and $f_n$ are homomorphisms; therefore $V$ is also an $H$-space.

**Theorem 5.** *For every finite complex $X$ there is an isomorphism $k^0(X; A) = [X, V]_*$, where $A = k^*(L_p^\infty)$.*

**Proof.** It is easy to verify that there is an epimorphism

$$\alpha : [X, V]_* \to \lim_{\leftarrow} [X, V^n] = k^0(X, A).$$

Introduce in the space $V$ the filtration $\{V_n\}$, putting

$$V_0 = V, \qquad V_n = \pi_n^{-1}(*), \quad * \in V^n.$$

Denote by $i_n : V_n \subset V$ and $i_{n,k} : V_n \subset V_k$ the natural embeddings. Clearly, all $V_n$ are fibrations in the sense of Serre.

$$V_n = \lim_{\leftarrow} V_n^k.$$

Let

$$\pi_n^k : V_n \to V_n^k$$

be the corresponding projections of fibrations.

**Lemma 3.** *Let $X$ be a cellular complex and let $\{\varphi_n : X \to V_n\}$ be a sequence of maps such that $\varphi_n \simeq i_{n+1,n} \circ \varphi_{n+1}$. Then the map $\varphi = \varphi_0 : X \to V$ is homotopic to a map to a point.*

Now let $X$ be a finite complex and $x \in \ker \alpha$. Put $\varphi_0 = x$, and, using the fact that for every finite complex the group $[X, \Omega^{2n} K_p]$ is finite, we can by induction construct a sequence of maps satisfying the lemma. The theorem is proved.

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*Note: Figure translations are in progress. See original paper for figures.*