THE MAXIMUM PRINCIPLE IN THE CLASS OF VARIATIONS SMALL IN ABSOLUTE VALUE, FOR OPTIMAL CONTROL PROBLEMS WITH MIXED CONSTRAINTS OF EQUALITY AND INEQUALITY TYPE

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 519.3

MATHEMATICS

A. Ya. Dubovitskii, A. A. Milyutin

THE MAXIMUM PRINCIPLE IN THE CLASS OF VARIATIONS SMALL IN ABSOLUTE VALUE, FOR OPTIMAL CONTROL PROBLEMS WITH MIXED CONSTRAINTS OF EQUALITY AND INEQUALITY TYPE

(Presented by Academician L. S. Pontryagin on 23 V 1969)

In this note the following problem is solved. In the class of variations small in absolute value, find necessary conditions for a minimum of the functional

$$I = \int_0^1 F(x(t), u(t), t)\, dt, \quad x \in E^n, \quad u \in E^r,$$

under the constraints

$$dx/dt = f(x, u, t), \quad x(0) = x_0, \quad x(1) = x_1, \quad g_k(x, u) = 0, \quad k = 1, \dots, k, \quad x, u \in Q_i,$$

$$i = 1, \dots, I.$$

Assumptions. $x(t), u(t)$ are bounded measurable functions.
$F(x, u, t), f(x, u, t)$: a) are bounded together with their first derivatives with respect to x, u on every bounded set of x, u, t; b) the derivatives with respect to x, u are uniformly continuous on every bounded set of x, u, t.

$g_k(x, u)$: a) are continuously differentiable; b) g_{ku} are linearly independent if $g_k(x, u) = 0$; c) $k < r$, Q_i is an open set, $\varkappa_i$ is the boundary of Q_i.

a) For every point $x, u \in \varkappa_i$ there exists an open convex cone $\Omega_i(x, u)$ in the space x, u such that:

1) if $\bar{x}, \bar{u} \in \overline{\Omega_i}(x, u)$, then $x + \varepsilon\bar{x},\ u + \varepsilon\bar{u} \in \bar{Q}_i$;

2) if $\bar{x}, \bar{u} \notin \overline{\Omega_i}(x, u)$, then $x + \varepsilon\bar{x},\ u + \varepsilon\bar{u} \notin \bar{Q}_i$

for all sufficiently small $\varepsilon > 0$ (the bar denotes closure); b) the set $x, u, n : x, u \in \varkappa_i,\ n \in N_i(x, u)$, where $N_i(x, u)$ is the set of exterior normals to $\Omega_i(x, u)$, is closed in the space x, u, n.

Let $x^0, u^0 \mid t$ be an optimal trajectory. If for arbitrary $\varphi_k(t), \xi$ there exist such $\bar{x}(t), \bar{u}(t)$ that

$$d\bar{x}/dt = f_x\bar{x} + f_u\bar{u}, \quad \bar{x}(0) = 0, \quad \bar{x}(1) = \xi,$$

$$g_{kx}\bar{x}(t) + g_{ku}\bar{u}(t) = \varphi_k(t)$$

(the derivatives are taken at the points $t;\ x^0u^0 \mid t$), then we shall say that the nondegeneracy condition is fulfilled for the trajectory. We shall assume that this condition is fulfilled for the trajectory $x^0, u^0 \mid t$.

Variations. In view of the nondegeneracy condition of the trajectory $x^0, u^0 \mid t$, the set L of variations $\bar{x}(t), \bar{u}(t)$ admissible with respect to the equality constraints is determined by the following conditions:

$$d\bar{x}/dt = f_x\bar{x} + f_u\bar{u}, \quad \bar{x}(0) = \bar{x}(1) = 0, \quad g_{kx}\bar{x} + g_{ku}\bar{u} = 0.$$

Put $n \in N_{i\delta}(x, u),\ x, u \in \bar{Q}_i$, if $n \in N_i(x', u'),\ x', u' \in \varkappa_i,\ \|x, u - x', u'\| \leq \delta$; $\bar{N}_{i\delta}(x, u)$ is the convex closure of $N_{i\delta}(x, u)$. Put $n, t \in \Pi_{i\delta}$, if $N_{i\delta}(x^0, u^0 \mid t) \neq \emptyset$ and $n \in \bar{N}_{i\delta}(x^0, u^0 \mid t)$.

If for some $\delta > 0$

$$n_x\bar{x}(t) + n_u\bar{u}(t) \leq -c < 0, \quad n, t \in \Pi_{i\delta},$$

then the variation $\bar{x}, \bar{u} \mid t$ is admissible with respect to the constraint $x, u \in \bar{Q}_i$, and conversely. The set of admissible variations forms a nonempty open cone Ω_i. The set Ω_0 of forbidden variations is defined by the inequality

$$\int_0^1 (F_x\bar{x} + F_u\bar{u})\, dt < 0.$$

The stationarity condition consists in the intersection $\Omega_0, L, \dots, \dots, \Omega_i, \dots$ being empty.

I. To translate this condition into the language of metric properties of the trajectory $x^0, u^0 \mid t$, we shall use Theorems 1 and 2 and the lemma of the present note.

Theorem 1. Let Z, X, Y be normed spaces, with $X = Y^*$, $Z = X^*$. In X a system of cones $\Omega_0, \dots, \Omega_n, \Omega$ is given, where $\Omega_0, \dots, \Omega_n$ are open convex cones, Ω is a convex cone; $x_0 \in \Omega_0, \dots, x_n \in \Omega_n$ is a certain system of elements.

Suppose that there are convex cones $\Omega^*_{iy} \subseteq \Omega^*_i$, $\Omega^*_y \subseteq \Omega^*$, weakly dense in $\Omega^*_0, \dots, \Omega^*$ and consisting of elements of the space Y^*.

Let B_ε be the set of systems $(y_0, \dots, y_n, y)$ such that $y_i \in \Omega^*_{iy}$,

$$y \in \Omega^*_y, \qquad \|y_0 + \dots + y_n + y\| \le \varepsilon, \qquad \sum_0^n y_i(x_i) = 1.$$

Let B be the set of systems $z_0, \dots, z_n, z$ that are solutions of the Euler equation relative to the system of cones $\Omega_0, \dots, \Omega_n, \Omega$ (i.e. $z_i \in \Omega^*_i$, $z \in \Omega^*$,

$$\sum_0^n z_i + z = 0$$

) such that

$$\sum_0^n z_i(x_i) = 1.$$

Then

$$B = \bigcap_{\varepsilon > 0} \overline{B}_\varepsilon$$

(closure in the weak sense).

In our case the space $X = L^{n+r}_\infty$. The space $Y = L^{n+r}_1$.

Lemma. Let $r_1(t), \dots, r_j(t), \dots$ be a sequence of measurable vector functions $(r \in E^n)$ possessing the following properties:

1) $\displaystyle\int |r_j| \, dt \le \text{const};$

2) for any natural $N \ge 0$ there exist:

a)

$$r^0_N(t) = \lim_j \operatorname{cl}(L_1) \, N r_j(t),$$

where

$$Nr = \begin{cases} r, & |r| \le N, \\ \dfrac{N}{|r|} r, & |r| > N; \end{cases}$$

b)

$$\lambda_N^0(t) = \lim_j \mathrm{cl}(L_1)\, N\lambda_j(t),$$

where

$$\lambda_j(t) = |r_j(t)|.$$

Put

$$\lambda^0(t) = \lim_{N\to\infty} \lambda_N^0(t), \qquad r^0(t) = \lim_{N\to\infty} r_N^0(t)$$

(the convergence takes place in L_1).

Then there exists an integer-valued function $N(j) \geq 0$ such that

1)

$$\lim_j \mathrm{cl}(L_\infty)\, N(j)\lambda_j(t) = \lambda^0(t),$$

$$\lim_j \mathrm{cl}(L_\infty)\, N(j)r_j(t) = r^0(t);$$

2) for any natural $N \geq 0$

$$\lim_j \int N(\lambda_j - N(j)\lambda_j)\, dt = 0.$$

With the aid of Theorem 1 and the lemma, the stationarity condition for the trajectory $x^0, n^0 \mid t$ can be written in the following equivalent form.

There exist $s_0, \psi(t), r_i(t), m_k(t)$, a measure ν, and a sequence $\delta_j, r_i^j(t), m_k^j(t)$ such that

$$1) \quad -s_0F_u + (\psi(t), f_u) + \sum_k m_k(t)g_{ku} - \sum_i r_{iu} = 0;$$

$$2) \quad s_0 \geq 0, \qquad \int \sum_k |m_k(t)|\, dt < +\infty, \qquad r_i(t) \in \operatorname{con} N_i(x^0, n^0 \mid t);$$

$$3) \quad -\frac{d\psi}{dt} = f_x^*\psi - s_0F_x + \sum_k m_k(t)g_{kx} - \sum_i r_{ix}(t) - \frac{d\nu}{dt};$$

4) the measure ν is characterized by the following properties:

$$\frac{d\nu}{dt} = \lim_j \mathrm{cl}(c)\left[\sum_i r^j_{ix}(t) - \sum_k m^j_k(t) g_{kx}\right],$$

$$r^j_i(t) \in \mathrm{con}\, N_{i\delta_j}(x^0, n^0 \mid t), \qquad \delta_j \to 0,$$

---

* In other words, it is necessary that from Ω^*_{ij}, $x \geq 0$ it follow that $x \in \overline{\Omega}_i$, and from Ω^*_y, $x \geq 0$ it follow that $x \in \overline{\Omega}$ (here the bar denotes closure).

$$\lim_j \int \left|\sum_k m^j_k g_{ku} - \sum_i r^j_{iu}\right| dt = 0;$$

5) **the normalization condition**

$$s_0 + \int_0^1 \sum_i |r_i|\, dt + \int_0^1 \sum_i |\dot{r}_i|\, dt = 1.$$

Thus the problem is reduced to describing the class of measures v, independent of the existence of the sequence $\delta_j, m^j_k(t), r^j_i(t)$. It is obvious that the class of measures v is determined only by the trajectory $x^0, u^0 \mid t$ and by the constraints $g_k(x,u) = 0$, $x, u \in \bar{Q}_i$, and does not depend on the form of the functional or of the differential relation.

This question is resolved by Theorem 2. Let R_0 be the set of $t, x, u, \tilde{r}_i, \beta_k$, where:

1) $x = x^0(t)$;

2) $u \in \overline{u^0(t)}$ *;

3) $\tilde{r}_i \in \mathrm{con}\, \bar{N}_i(x,u)$, $x, u \in \varkappa_i$; $\tilde{r}_i = 0$, $x, u \notin \varkappa_i$, $\sum_i |\tilde{r}_i| = 1$;

4)
$$\sum_k \beta_k g_{ku} - \sum_i \tilde{r}_{iu} = 0.$$

We shall call the set R_0 the set of phase points of the trajectory $x^0, u^0 \mid t$.

Theorem 2. 1) For any continuous function $\xi(t)$ ($\xi \in E^n$), $\int \xi\, dv$ admits the representation

$$\int \xi\, dv = \int_{R_0} \left(\sum_i \tilde{r}_{ix} - \sum_k \beta_k g_{ku},\, \xi(t)\right) d\sigma,$$

$$d\sigma \geq 0, \qquad \int_{R_0} d\sigma = \sum_i \int |\dot{r}^j_i|\, dt;$$

2) for any measure $\sigma \geq 0$ concentrated on R_0,

$$\int_{R_0} \sum_i \tilde{r}_{ix} - \sum_k \beta_k g_{kx},\, \xi\, d\sigma$$

admits the representation

$$\int_{R_0} \left(\sum_i \tilde{r}_{ix} - \sum_k \beta_k g_{kx},\, \xi \right) d\sigma = c \int \xi\, dv, \quad \text{where } c > 0,$$

$$\int_{R_0} d\sigma = \sum_i \int |\dot{r}_i^j|\, dt.$$

Using the representation of the measure v, we obtain the final form of the stationarity condition.

II. The Euler equation of the problem under investigation has the form

$$0 = -s_0 \int (F_x x' + F_u \bar{u})\, dt - c, x'(1) + \sum_k m_k (g_{kx} x' + g_{ku} \bar{u}) - \sum_i \lambda_i (n'_{xx} + n_u \bar{u}), \tag{B}$$

$$s_0 \geq 0, \qquad m_k \in r^*_{\infty(t)}, \qquad \lambda_i \in r^*_{\infty(n,t)},$$

$$\lambda_i \geq 0, \qquad \Pi_i \lambda_i = \lambda_i^* **, \qquad s_0 + \sum_i \|\lambda_i\| > 0,$$

$$dx'/dt = f'_{xx} + f_u \bar{u}, \qquad x'(0) = 0.$$

This condition is equivalent to the one found. Let us clarify how they are related. We decompose, in accordance with (3), m_k and λ_i into the sum of absolutely continuous and singular components. We obtain:

$$-s_0 \int (F'_{xx} + F_u \bar{u})\, dt + c, x'(1) + \sum_k \int m_k(t)(g_{kx} x' + g_{ku} \bar{u})\, dt -$$

$$- \sum_i \int (r_{ix} x' + r_{iu} \bar{u})\, dt + \sum_k m_k^s (g_{kx} x') - \sum_i \lambda_i^s (n'_{xx}) =$$

* $\overline{u^0}(t)$ denotes the closure of the graph of $u^0(t)$ with respect to the measure (see (3)).

** $\Pi_i \lambda_i(z(n,t)) = \lim_{\delta \to 0} \lambda_i(\chi_{\Pi_i^\delta} z(n,t))$ (see (3)).

$$= - \sum_k m_k^s (g_{ku}(\bar{u})) + \sum_i \lambda_i^s (n_u \bar{u}), \quad \text{where } \sum_k \int |m_k(t)|\, dt < +\infty, \quad r_i(t) \in$$

$$\in \operatorname{con} N_i(x^0, u^0 \mid t), \quad 0 < s_0 + \sum_i \int |r_i|\, dt + \sum_i \|\lambda_i^s\| < +\infty, \quad m_k^s \in$$

$$\in L_\infty^*(t), \quad \lambda_i^s \in L_\infty(n,t) \text{ are singular components.}$$

Since the left-hand side of the equality is an absolutely continuous functional of $\bar{u}$, while the right-hand side is a singular functional of $\bar{u}$, they are both equal to zero.

Since $x'(t)$ is a continuous function, the problem arises of describing the class of measures μ admitting the representation:

$$-\int \xi \, d\mu = \sum_k m_k^s(g_{kx}x') - \sum_i \lambda_i^s(n_x x'), \tag{α}$$

where

$$\sum_k m_k^s(g_{ku}\bar{u}) - \sum_i \lambda_i^s(n_u \bar{u}) = 0, \quad \sum_i \|\lambda_i^s\| > 0, \quad \lambda_i^s \geq 0, \quad \Pi_i \lambda_i^s = \lambda_i^s. \tag{β}$$

This problem is solved as follows. In the space $L_\infty : z_i(n,t), \bar{u}(t)$ consider the subspace $\mathcal{L} : g_{ku}\bar{u} = 0,\ z_i = n_u \bar{u}$ and the cones $\widetilde{\Omega}_i :\ \lim_{\delta\to 0} \operatorname{vrai\,min} z_i(n,t) > 0,\ n,t \in \Pi_{i\delta}$. Condition (β) holds if and only if the system $\mathcal{L}, \dots, \widetilde{\Omega}_i, \dots$ has empty intersection. Since the space $L_\infty = L_1^*$, we may apply Theorem 1.

As a result we obtain: in order that $-\int \xi \, d\mu$ admit the representation $(\alpha), (\beta)$, it is necessary and sufficient that:

$$\frac{d\mu}{dt} = \lim_j \operatorname{cl}(c) \left[\sum_i r_{ix}^j(t) - \sum_k m_k^j(t) g_{kx} \right],$$

where

$$r_i^j(t) \in \operatorname{con} \widetilde{N}_{i\delta_j}(x_i^0, u^0 \mid t), \quad \delta_j \to 0, \quad c_1 \leq \int \sum_i |r_i^j| \, dt \leq c_2,$$

$$c_1, c_2 > 0, \quad \lim_j \int \left| \sum_k m_k^j g_{ku} - \sum_i r_{iu}^j \right| dt = 0.$$

Thus we see that the class of measures μ coincides with the class of measures ν considered in Part I of the note.

In conclusion, we note:

1. The results of [3] are contained in those presented here, for in [3] smooth inequality constraints were considered and there were no constraints of the form $g_k(x,u) = 0$.

2. In this note, by means of the concept of the set of phase points of a trajectory, both purely phase and mixed constraints are considered from a unified point of view. Until now, in the literature, mixed constraints have been subjected to a regularity requirement, i.e., the requirement that there be no phase points generated by the mixed constraints.

Institute of Chemical Physics
Academy of Sciences of the USSR
Moscow

Received
4 IV 1969

REFERENCES

1. L. S. Pontryagin, V. G. Boltyanskii et al., *Mathematical Theory of Optimal Processes*, Moscow, 1961.

2. A. Ya. Dubovitskii, A. A. Milyutin, *Zh. Vychisl. Matem. i Matem. Fiz.*, **5**, No. 3, 395 (1965).

3. A. Ya. Dubovitskii, A. A. Milyutin, ibid., **8**, No. 4, 725 (1968).

Note: Figure translations are in progress. See original paper for figures.