ON THE THEORY OF THE GENERALIZED CARLEMAN BOUNDARY-VALUE PROBLEM

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Abstract

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MATHEMATICS

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ON THE THEORY OF THE GENERALIZED CARLEMAN BOUNDARY-VALUE PROBLEM

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Let a closed Lyapunov contour L bound a finite simply connected domain D^+. The complement of $D^+ + L$ to the full plane will be denoted by D^-. Consider the following boundary-value problem: to find a function $\Phi^+(z)$, analytic in D^+ and H-continuous in $D^+ + L$, subject to the boundary condition

$$\Phi^+[\alpha(t)] = a(t)\Phi^+(t) + b(t)\overline{\Phi^+(t)} + h(t) \tag{1}$$

or

$$\Phi^+[\alpha(t)] = a(t)\Phi^+(t) + b(t)\overline{\Phi^+(t)}, \tag{1'}$$

where $\alpha(t)$ maps the contour L homeomorphically onto itself, preserving ($\alpha = \alpha_+(t)$) or reversing ($\alpha = \alpha_-(t)$) the direction of traversal on L; the Carleman condition $\alpha[\alpha(t)] \equiv t$, $\alpha'(t) \neq 0$, $\alpha'(t) \in H(L)$ is satisfied; the H-continuous functions $a(t)$, $b(t)$, and $h(t)$ satisfy the conditions

$$a(t)a[\alpha(t)] + \overline{b(t)}b[\alpha(t)] = 1, \tag{2}$$

$$a(t)b[\alpha(t)] + \overline{a[\alpha(t)]}b(t) = 0, \tag{2'}$$

$$a(t)h[\alpha(t)] + b(t)\overline{h[\alpha(t)]} + h(t) = 0. \tag{2''}$$

The problem formulated generalizes the Carleman problem posed by Carleman ([2]) and studied by D. A. Kveselava ([3]), which is obtained from (1) when $\alpha = \alpha_-(t)$, $b(t) \equiv 0$, and the Carleman-type problem posed and studied in ([4]) by one of the authors of this article and E. G. Khasabov, which is obtained from (1) when $\alpha = \alpha_+(t)$, $a(t) \equiv 0$.

The initiative in posing problem (1) belongs to N. P. Vekua. In his work ([5]), in the case $\alpha = \alpha_-(t)$, a regularization algorithm was constructed which makes it possible to obtain solutions of problem (1) for a system of several functions from solutions of a system of Fredholm-type integral equations. However, important questions concerning the magnitude of the index and the conditions for the normal solvability of problem (1) remained unresolved, and, even more so, the question of the number of solutions and the conditions for solvability of this problem. The present article fills this gap in the theory of problem (1).

Solutions of the boundary-value problems (1) and ($1'$), obviously, can be obtained from solutions of the problems

$$\Phi_1^+[\alpha(t)] = a(t)\Phi_1^+(t) + b(t)\overline{\Phi_2^+(t)} + h(t), \tag{3}$$

$$\Phi_1^+[\alpha(t)] = a(t)\Phi_1^+(t) + b(t)\overline{\Phi_2^+(t)} \tag{3'}$$

for two functions $\Phi_1^+(z)$ and $\Phi_2^+(z)$, analytic in the domain D^+, by requiring that $\Phi_1^+(z) \equiv \Phi_2^+(z)$. In doing so we assume that the functions $a(t)$, $b(t)$, $h(t)$, $\alpha(t)$, and $\alpha'(t)$ satisfy all the conditions stated above and, in addition,

$$b(t) \neq 0 \quad \text{on } L. \tag{4}$$

The relation between the solutions of problems ($1'$) and ($3'$) is established by the following

Lemma 1. *Under conditions (2), ($2'$), and (4), the general solution of the boundary-value problem ($1'$) contains half as many arbitrary real constants as the general solution of problem ($3'$).*

The assertion of the lemma follows from the fact that, when conditions (2), ($2'$), and (4) are satisfied, the fundamental system of solutions of problem ($3'$) can be chosen in the form of pairs of functions with equal components.

The investigation of problem (3) is based on the following lemmas.

Lemma 2. If $a = a_+(t)$, $\alpha[\alpha(t)] \equiv t$, the functions $\Phi_1^+(z)$ and $\Phi_2^+(z)$ are analytic in D^+ and H-continuous in $D^+ + L$, then there exist a function $\varphi(t)$ and a complex constant C, determined uniquely, such that the integral representation is valid

$$\Phi_1^+(z) = \frac{1}{2\pi i}\int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\,d\tau + C,$$

$$\Phi_2^+(z) = -\frac{1}{2\pi i}\int_L \frac{\overline{\varphi(\tau)}}{\tau - z}\,d\tau + \overline{C}. \tag{5}$$

Lemma 3. If $a = a_-(t)$, $\alpha[\alpha(t)] \equiv t$, the functions $\Phi_1^+(z)$ and $\Phi_2^+(z)$ are analytic in D^+ and H-continuous in $D^+ + L$, then one can uniquely determine the density $\varphi(t)$ and the complex constant C such that

$$\Phi_1^+(z) = \frac{1}{2\pi i}\int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\,d\tau + C,$$

$$\Phi_2^+(z) = -\frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\,d\tau + C. \tag{5'}$$

Using (5), if $a = a_+(t)$, or (5′), if $a = a_-(t)$, the Sokhotski formulas and the Carleman condition, we reduce the auxiliary problem (3) to equivalent integral equations for $\varphi(t)$:

for $a = a_+(t)$

$$[1+b(t)]\varphi(t) - a(t)\varphi[\alpha(t)] + \frac{1}{\pi i}\int_L \left[\frac{a'(\tau)}{\alpha(\tau)-\alpha(t)} - \frac{b(t)\overline{t'^2(\sigma)}}{\tau - t}\right]\varphi(\tau)\,d\tau - \frac{a(t)}{\pi i}\int_L \frac{\varphi[\alpha(\tau)]}{\tau - t}\,d\tau = 2C[\alpha(t)+b(t)-1]+2 \tag{6}$$

in the case $a = a_-(t)$

$$\varphi(t) - a(t)\varphi[\alpha(t)] + b(t)\overline{\varphi(t)} - \frac{1}{\pi i}\int_L \frac{\varphi(\tau)\alpha'(\tau)}{\alpha(\tau)-\alpha(t)}\,d\tau - \frac{a(t)}{\pi i}\int_L \frac{\varphi[\alpha(\tau)]}{\tau - t}\,d\tau + b(t)\overline{\frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\,d\tau} = 2\{[a(t)-1]C + b(t)\overline{C} \tag{6'}$$

Relying on the Noether theory of integral equations in (6) and (6′), and also using Lemma 1, we obtain the following assertions:

Theorem 1. If $a = a_+(t)$, $\alpha[\alpha(t)] \equiv t$, conditions (2)–(2″) and (4) are satisfied, then the boundary-value problems (1) and (3) are Noetherian with index $I = \operatorname{ind} b(t) + 1$. For the solvability of problems (1) and (3) it is necessary and sufficient that the conditions

$$\operatorname{Re}\int_L \Psi^+(t)\frac{\overline{h(t)}}{\overline{b(t)}}\,dt = 0 \qquad \text{in the case of problem (1),}$$

$$\int_L \Psi_2^+(t)\frac{\overline{h(t)}}{\overline{b(t)}}\,dt = 0 \qquad \text{in the case of problem (3),}$$

be fulfilled, where $\Psi^+(z)$ and $\Psi_2^+(z)$ are arbitrary solutions of the corresponding adjoint problems

$$\alpha'(t)\Psi^+[\alpha(t)] = -a[\alpha(t)]\Psi^+(t) + \overline{b[\alpha(t)]}\,{t'}^2(s)\Psi^+(t),$$

$$\alpha'(t)\Psi_1^+[\alpha(t)] = -a[\alpha(t)]\Psi_1^+(t) + \overline{b[\alpha(t)]}\,{t'}^2(s)\Psi_2^+(t).$$

Theorem 2. If $\alpha = \alpha_-(t)$, $\alpha[\alpha(t)] \equiv t$, conditions (2)–(2″) are satisfied and the inequality $a(t)b(t) \neq 0$ holds, then problems (1) and (3) are of non-Noetherian type with index $I = -\operatorname{ind} a(t) + 1$. For solvability of problems (1) and (3), it is necessary and sufficient that the conditions

$$\operatorname{Re}\int_L \Psi^+(t)\frac{\overline{h(t)}}{\overline{b(t)}}\,dt = 0 \quad \text{in the case of problem (1)},$$

$$\int_L \Psi_2^+(t)\frac{\overline{h(t)}}{b(t)}\,dt = 0 \quad \text{in the case of problem (3)},$$

be fulfilled, where $\Psi^+(z)$ and $\Psi_2^+(z)$ are arbitrary solutions of the corresponding adjoint problems

$$\alpha'(t)\Psi^+[\alpha(t)] = a[\alpha(t)]\Psi^+(t) + \overline{b[\alpha(t)]{t'}^2(s)\Psi^+(t)},$$

$$\alpha'(t)\Psi_1^+[\alpha(t)] = a[\alpha(t)]\Psi_1^+(t) + \overline{b[\alpha(t)]{t'}^2(s)\Psi_2^+(t)}.$$

Using the results of (7) and Theorem 1, we obtain the following assertion on the solvability of problem (1).

Theorem 3. If $\alpha = \alpha_+(t)$, $\alpha[\alpha(t)] \equiv t$, conditions (2)–(2″) are satisfied and the condition $|b(t)| > |a(t)|$ holds, then the boundary-value problem (1) has $l = \max[0, \operatorname{ind} b(t) + 1]$ linearly independent solutions. The nonhomogeneous problem (1) is solvable provided $p = \max[0, -\operatorname{ind} b(t) - 1]$ solvability conditions are fulfilled.

Under the assumptions made concerning the functions $a(t)$, $b(t)$, $h(t)$, $\alpha(t)$, and $\alpha'(t)$, let us consider the boundary-value problems

$$\Phi_1^+[\alpha(t)] = a(t)\Phi_2^+(t) + b(t)\overline{\Phi_2^+(t)} + h(t), \tag{7}$$

$$\Phi_1^+[\alpha(t)] = a(t)\Phi_2^+(t) + b(t)\overline{\Phi_2^+(t)}, \tag{7'}$$

$$\Phi_*^+[\alpha(t)] = -a(t)\Phi_*^+(t) - b(t)\overline{\Phi_*^+(t)}. \tag{8}$$

Lemma 3. The equality

$$\tilde{l} = l + l_*,$$

holds, where l, l_*, and $\tilde{l}$ are, respectively, the numbers of linearly independent solutions of the boundary-value problems (1), (8), and (7′).

Let us formulate two theorems obtained for problem (7).

Theorem 4. Suppose $\alpha = \alpha_-(t)$, $\alpha[\alpha(t)] \equiv t$, conditions (2)–(2″) are satisfied and $a(t) \neq 0$. Then the boundary-value problem (7) belongs to the Noetherian type, the index of the problem is equal to $I = 2[-\operatorname{ind} a(t) + 1]$, and for solvability of problem (7) the following conditions are necessary and sufficient:

$$\operatorname{Re} \int_L h[\alpha(t)] \Psi_2^+(t)\, dt = 0,$$

where $\Psi_2^+(z)$ is the second component of an arbitrary solution of the adjoint problem

$$\alpha'(t)\Psi_1^+[\alpha(t)] = a[\alpha(t)]\Psi_2^+(t) + \overline{b[\alpha(t)]t'^2(s)\Psi_2^+(t)}.$$

Theorem 5. If $\alpha = \alpha_-(t)$, $\alpha[\alpha(t)] \equiv t$, conditions (2)–(2″) are satisfied and $|a(t)| > |b(t)|$, then problem (7′) has $\tilde{l} = \max\{0, 2[-\operatorname{ind} a(t) + 1]\}$ linearly independent solutions, while problem (7) is solvable provided $\tilde{p} = \max\{0, 2[\operatorname{ind} a(t) - 1]\}$ solvability conditions are fulfilled.

The results of the investigation of the auxiliary problem (7) make it possible to obtain the following proposition on the solvability of problem (1).

Theorem 6. If $\alpha = \alpha_-(t)$, $\alpha[\alpha(t)] \equiv t$, conditions (2)–(2″) are satisfied and the inequality $|a(t)| > |b(t)|$ holds, then problem (1′) has $l = \max[0, -\operatorname{ind} a(t) + 2 - \lambda]$ linearly independent solutions, and for solvability of problem (1) it is necessary and sufficient that $p = \max[0, \operatorname{ind} a(t) -$

$-2 + \lambda$ solvability conditions, where $\lambda = 0$ if $a(t_*) = a(t_{**}) = 1$; $\lambda = 1$ if $a(t_*) = -a(t_{**}) = \pm 1$; $\lambda = 2$ if $a(t_*) = a(t_{**}) = -1$; t_*, t_{**} are fixed points of the transformation $\alpha_-(t)$.

In particular, for $b(t) \equiv 0$ we obtain the well-known theorem of D. A. Kveselava on the solvability of the Carleman problem.

Problem (1) can be put into correspondence with a certain Riemann boundary-value problem for two pairs of unknown functions (see, for example, (8)). We shall call problem (1) stable if the partial indices of the corresponding Riemann problem are stable.

Theorem 7. Problem (1) is stable if one of the following conditions is satisfied:

1) $\alpha = \alpha_+(t),\ |b(t)| > |a(t)|;$

2) $\alpha = \alpha_-(t),\ |a(t)| > |b(t)|.$

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Note: Figure translations are in progress. See original paper for figures.