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Abstract

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ELLIPTIC PSEUDODIFFERENTIAL OPERATORS ON A CLOSED MANIFOLD DEGENERATING ON A SUBMANIFOLD

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Let on a smooth compact manifold M of dimension n there be given a pseudodifferential operator $P(x,D)$ of order $m>0$. It is assumed that $P(x,D)$ satisfies the ellipticity condition for $x\in M\setminus\Gamma$, where Γ is a smooth submanifold of dimension $n-1$. Let a covering $\{U_i\}$ of some neighborhood of the submanifold Γ be fixed, and in each neighborhood U_i let a local coordinate system (l.c.s.) $(x_1,\dots,x_n)$ be so defined that Γ is given by the equation $x_n=0$, and, for any intersecting U_i and U_j, the passage from the l.c.s. $(x_1^i,\dots,x_n^i)$ in U_i to the l.c.s. $(x_1^j,\dots,x_n^j)$ in U_j is effected by means of a transformation of the form

$$x_1^j=\varphi_1(x_1^i,\dots,x_{n-1}^i),\dots,\quad x_{n-1}^j=\varphi_{n-1}(x_1^i,\dots,x_{n-1}^i),$$

$$x_n^j=\varphi_n(x_1^i,\dots,x_n^i).$$

We note that such transformations preserve the coordinate lines $x'=\text{const}$, where $x'=(x_1,\dots,x_{n-1})$. Denote by ω the manifold $x_n=0$, $\xi_n=0$, $\xi'\neq 0$ in the cotangent bundle of the manifold M. Let, in the l.c.s. corresponding to U_i, the full symbol $p(x,\xi)\sim\sum_{j=0}^{\infty}p_j(x,\xi)$ of the operator $P(x,D)$ possess the properties:

1. There is given a $\delta>0$ such that $m\delta$ is an integer and $p_0(x,\xi)=0$ on ω, $p_{0\beta_n}^{\alpha_n}(x,\xi)=0$ on ω for $\beta_n<(m-\alpha_n)\delta$, $p_{j\beta_n}^{\alpha_n}(x,\xi)=0$ on ω for $\beta_n<(m-j-\alpha_n)\delta-j$, where

$$p_{j\beta_n}^{\alpha_n}(x,\xi)=\frac{\partial^{\alpha_n+\beta_n}}{\partial x_n^{\beta_n}\partial\xi_n^{\alpha_n}}p_j(x,\xi)\ *.$$

2. $p_0(x,\xi)\neq 0$ outside ω and

$$\frac{\partial^m}{\partial \xi_n^m} p_0(x,\xi) \neq 0$$

on ω.

3. For $x_n \neq 0$ and $\xi \neq 0$

$$L_0^0(x', x_n, \xi) = \sum_{\alpha_n=0}^{m} \frac{1}{\alpha_n! \beta_n!} x_n^{\beta_n} p_{0\beta_n}^{\alpha_n}(x', 0, \xi', 0) \xi_n^{\alpha_n} \neq 0, \qquad \beta_n = (m-\alpha_n)\delta, \tag{1}$$

where the summation is carried out only over those indices for which $\beta_n = (m-\alpha_n)\delta$ is an integer.

An example of a symbol with the indicated properties is $p_0(x,\xi) = i\xi_n + a x_n^r |\xi|$, where $\operatorname{Re} a \neq 0$, r is an integer. Operators with such symbols arise in the study of the oblique derivative problem for the Laplace equation. It is known that such operators may have an infinite-dimensional kernel (cokernel).

* Here $p_j(x,\xi)$ are functions of class C^∞ for $\xi \neq 0$ and homogeneous in ξ of order $m-j$. For the definition of pseudodifferential operators see (1).

In the present article the operator $P(x,D)$ is studied in natural functional spaces. It follows from the theorems proved below that, generally speaking, it has an infinite-dimensional kernel and cokernel. However, if additional boundary and coboundary conditions are prescribed on Γ, then the corresponding problem becomes Noetherian (i.e., has finite-dimensional kernel and cokernel). For such problems a right and a left regularizer are constructed below. We note that an analogous question in the case of differential operators degenerating on the boundary of a domain was considered in ([2]).

Lemma. If the complete symbol $p(x,\xi)$ of the operator $P(x,D)$ in each neighborhood U_i satisfies conditions 1–3, then $p_0(x,\xi)$ can be represented in the form

$$p_0(x,\xi) = q_{m0}^0(x,\xi)\xi_n^m + \sum_{\alpha_n=0}^{m-1} \sum_{m\delta \geq \beta_n \geq (m-\alpha_n)\delta} x_n^{\beta_n} q_{\alpha_n\beta_n}^0(x,\xi)\xi_n^{\alpha_n}, \tag{2}$$

where $q_{\alpha_n\beta_n}^0(x,\xi)$ are homogeneous functions in ξ of orders $m-\alpha_n$ of class C^∞ for $\xi \neq 0$, and β_n are integers. The terms $p_j(x,\xi)$ of the complete symbol of the operator $P(x,D)$ admit an analogous expansion

$$p_j(x,\xi) = q_{j0}^j(x,\xi)\xi_n^j + \sum_{\alpha_n=0}^{m-j-1} \sum_{m\delta \geq \beta_n \geq (m-j-\alpha_n)\delta - j} x_n^{\beta_n} q_{\alpha_n\beta_n}^j(x,\xi)\xi_n^{\alpha_n}, \tag{3}$$

where $q^j_{\alpha_n\beta_n}(x,\xi)$ are homogeneous functions in ξ of order $m-j-\alpha_n$ of class C^∞ for $\xi \neq 0$. For sufficiently small x_n the symbol $q^0_{m0}(x,\xi)$ is elliptic, i.e. for $\xi \neq 0$, $\xi \in R^n$, $q^0_{m0}(x,\xi) \neq 0$.

As in ([2]), the functional space $H_{(m,\delta)}(M,\Gamma)$ is introduced, with norm $\|\cdot\|_{m,\delta}$, equivalent to the usual norm of the space $H_m(M)$ outside a neighborhood of Γ, and for $u(x)$ with support in a neighborhood of Γ

$$\|u\|_{m,\delta} = \sum_i \sum_{\alpha_n=0}^{m} \left\| x_n^{(m-\alpha_n)\delta} (\Lambda')^{m-\alpha_n} D_n^{\alpha_n} \varphi_i u \right\| + \|u\|,$$

where Λ' is the operator with symbol $1+|\xi'|$; $\|\cdot\|$ is the norm in $L_2(M)$; φ_i is a partition of unity in a neighborhood of Γ, subordinate to the covering U_i. We note that $H_{(m,\delta)}(M,\Gamma)$ is embedded in $H_{m/(1+\delta)}(M)$.

Suppose that the characteristic equation

$$L^0_0(x', x_n, \xi', \zeta) = 0, \tag{4}$$

for $x_n > 0$ and $\xi' \neq 0$ has μ roots with $\operatorname{Im}\zeta > 0$, and for $x_n < 0$ has ν roots with $\operatorname{Im}\zeta < 0$. Consider the following problem on M:

$$P(x,D)u + \sum_{i=1}^{k} G_i \rho_i(x') \otimes \delta(\Gamma) = f(x), \tag{5}$$

$$\gamma B_j u + \sum_{i=1}^{k} E_{ji}\rho_i(x') = g_j(x'), \qquad 1 \le j \le l, \tag{6}$$

where γ is the operator of restriction of functions to Γ; E_{ji} are pseudodifferential operators on Γ (with homogeneous symbols $e_{ji}(x',\xi')$); B_j and G_i are pseudodifferential operators on M, and in a neighborhood of Γ the symbols $b_j(x,\xi)$ for B_j are quasihomogeneous of order m_j in ξ, i.e. $b_j(x, \lambda^{1+\delta}\xi', \lambda\xi_n) = \lambda^{m_j} b_j(x,\xi)$, $\lambda > 0$, while the symbols $g_i(x,\xi)$ for G_i are quasihomogeneous in ξ of order σ_i. For nondegenerate elliptic operators such problems were considered in ([3,4]).

The number k of coboundary operators in (5) and the number l of boundary operators in (6) must be related to the numbers μ and ν of roots of equation (4) by the formula

$$\mu + \nu - m = l - k. \tag{7}$$

It is assumed that $\sigma_i < -1/2$, $m_j < m - 1/2$, and the order of E_{ji} is equal to $t_i - s_j$, where $s_j = (m - m_j - 1/2)/(1+\delta)$, $t_i = (\sigma_i + 1/2)/(1+\delta)$.

Theorem 1. The operator $\mathfrak{A}$:

$$(u, \rho_1, \dots, \rho_k) \xrightarrow{\mathfrak{A}} (f, g_1, \dots, g_l),$$

defined by (5), (6), gives a continuous mapping

$$\mathcal{H}_1 \equiv H_{(m,\delta)}(M, \Gamma) \times H_{t_1}(\Gamma) \times \cdots \times H_{t_k}(\Gamma) \xrightarrow{\mathfrak{A}}$$

$$\xrightarrow{\mathfrak{A}} L_2(M) \times H_{s_1}(\Gamma) \times \cdots \times H_{s_l}(\Gamma) \equiv \mathcal{H}_2.$$

In order to formulate the conditions for normal solvability of problem (5), (6), consider on the line $\mathbb{R}^1$ the problem

$$\frac{1}{q^0_{m0}(x', 0, \xi', 0)} \Bigg(L^0_0(x', x_n, \xi', D_n) v(x_n) + \sum_{\tau > j \geq 1} L^0_j(x', x_n, \xi', D_n) v(x_n)$$

$$+ \sum_{i=1}^{k} \tilde{g}_i(x', 0, \xi', x_n) \rho_i = f_1(x_n), \tag{8}$$

$$(\tilde{b}_j(x', 0, \xi', x_n), v(x_n)) + \sum_{i=1}^{k} e_{ji}(x', \xi') \rho_i = \psi_j, \tag{9}$$

where

$$\tau = \left[\frac{m\delta}{1+\delta} \right],$$

$\tilde{g}_i(x', 0, \xi', x_n)$ and $\tilde{b}_j(x', 0, \xi', x_n)$ are the inverse Fourier transforms of $g_i(x', 0, \xi)$ and $b_j(x', 0, \xi)$ with respect to ξ_n; $L^0_j(x', x_n, \xi)$ is given by a formula analogous to (1), in which p_0 is replaced by p_j, $\beta_n = (m - \alpha_n)\delta$ is replaced by $\beta_n = (m - j - \alpha_n)\delta - j$, and the summation over α_n is carried out up to $\alpha_n = m - j$. The following is assumed to hold.

Condition $Z_{\xi'}$. The problem (8), (9) with zero right-hand sides f_1 and ψ_j, for any $\xi' \neq 0$, $\xi' \in \mathbb{R}^{n-1}$, has only the trivial solution in the class of functions with finite norm

$$\|v(x_n)\|^2_{m,\delta} = \sum_{j=0}^{m} \int (1 + |x_n|)^{2(m-j)\delta} |D^j_n v(x_n)|^2 \, dx_n. \tag{10}$$

Theorem 2. If conditions 1-3 are satisfied and at each point $x' \in \Gamma$ condition $Z_{\xi'}$ is satisfied, then the operator $\mathfrak{A}$ is Noetherian.

For the proof of the theorem it is sufficient to construct a regularizer, i.e. such a continuous operator R from $\mathcal{H}_2$ to $\mathcal{H}_1$ that $\mathfrak{A}R = I + T_1$, $R\mathfrak{A} = I + T_2$, where I is the identity operator and T_1 and T_2 are completely continuous. Since the operator $P(x, D)$ is elliptic outside Γ, by means of a partition of unity the construction of R reduces to the construction of a regularizer of the operator $\mathfrak{A}$ in a neighborhood of a point $x'_0 \in \Gamma$. Multiplying equality (5) by the operator with symbol

$$\frac{1}{q^0_{m0}(x,\xi)},$$

we obtain the equation

$$P_1(x,D)u + \sum_{i=1}^{k} H_i \rho_i(x') \otimes \delta(x_n) = f_1(x). \tag{11}$$

It follows from the composition formula that the operator $P_1(x,D)$ also satisfies conditions 1-3, and for it $q^0_{m0}(x,\xi)$ in the expansion (2) is equal to 1. Next, the operation of freezing the coefficients of the operators B_j, E_{ji} is performed, i.e. they are replaced by operators with symbols $b_j(x'_0,0,\xi)$, $e_{ji}(x'_0,\xi')$. The freezing operation for $P_1(x,D)$ is somewhat more complicated. The operator $P_1(x,D)$ is replaced by an operator with symbol in which $q^j_{\alpha\beta_n}(x,\xi)$ in (2) and (3) are replaced by $q^j_{\alpha\beta_n}(x'_0,0,\xi',0)$, and the terms with

$$\beta_n > (m-j-\alpha_n)\delta - j$$

are discarded. The operator H_i is replaced by an operator with symbol

$$\frac{g_i(x',0,\xi)}{q^0_{m0}(x',0,\xi',0)}.$$

After the Fourier transform with respect to x', we arrive at problem (8), (9), where f_1, ψ_j, ρ_i, and v depend on ξ'.

As shown in ([2]), for $\xi' \neq 0$ on the half-axis $x^n > 0$, equation (8), whose boundary terms have been moved to the right-hand side, has, for arbitrary ρ_i and $f_1(x_n) \in L_2(R^1_+)$, a particular solution $v(x_n)$ with finite norm (10), where the integral is taken over the half-axis; and the homogeneous equation has a μ-parameter family of solutions with the same finite norm. Similarly, for $x_n < 0$ we obtain a ν-parameter family of solutions. Writing out the matching conditions up to order $m-1$ for these two families at $x_n = 0$, and using the boundary conditions, we obtain $m+l$ equations for the arbitrary constants and the unknown densities ρ_i. In view of condition (7), this is a square system of linear equations, which is uniquely solvable, since condition $Z_{\xi'}$ is satisfied. On the basis of this solution one constructs a regularizer R_0 for problem (8), (9), analogously to how this was done in ([2]).

With the aid of R_0, a regularizer for problem (11), (6) is constructed in the usual way. Here the essential point is that the symbol

$$x_n^{\beta_n}[q^j_{\alpha_n\beta_n}(x,\xi) - q^j_{\alpha_n\beta_n}(x,\xi',0)]\xi_n^{\alpha_n}$$

in a sufficiently small neighborhood of the point x_0' corresponds either to a completely continuous operator or to an operator with small norm. To prove the latter assertion, note that if $q(x,\xi)$ is a homogeneous function in ξ of order $r > 0$, of class C^∞ for $\xi \neq 0$, then

$$q(x,\xi) - q(x,\xi',0) = Q(x,\xi)\xi_n^r + \sum_{\alpha_n=1}^{r-1} \frac{1}{\alpha_n!} q^{\alpha_n}(x,\xi',0)\xi_n^{\alpha_n}, \tag{12}$$

where $Q(x,\xi)$, for $\xi \neq 0$, is continuous together with all derivatives with respect to the variables x, and the order of homogeneity of the function $Q(x,\xi)$ in ξ is equal to zero. Therefore $Q(x,\xi)$ serves as the symbol of a bounded operator in $L_2(M)$. Applying expansion (12) to the functions $q^j_{\alpha_n\beta_n}(x,\xi)$ and passing to the corresponding operators, we obtain that our assertion is a consequence of the embedding theorems established in ([2]). Theorem 1 is proved by analogous arguments.

Let us note that if the right-hand sides $f(x)$ and $g_j(x')$ have additional derivatives in the tangential directions, then the solutions $u(x)$ and $\rho_i(x')$ also have additional smoothness with respect to the tangential variables (the proof is analogous to ([2])). If $f \in C^\infty(M)$, $g_j \in C^\infty(\Gamma)$, then in some cases it follows from this that $u \in C^\infty(M)$; for example, this is true if there are no boundary operators. By analogous methods one proves

Theorem 3. *Suppose that conditions 1-3 are satisfied. If equation (8) for $\xi' \neq 0$, with $f = 0$ and $\rho_i = 0$, has only the trivial solution with finite norm (10), then the operator $P(x,D)$ is hypoelliptic.*

We note that, as was proved in ([2]), solutions of equation (8) with $f = 0$ and $\rho_i = 0$ that have finite norm (10) decay at infinity, together with all derivatives, faster than any power.

Remark 1. Above we considered the case in which the symbol has, on Γ, the same order of degeneracy with respect to all variables $\xi_1, \dots, \xi_{n-1}$. Analogously to ([2]), one may study the case in which the orders of degeneracy with respect to the different variables $\xi_1, \dots, \xi_{n-1}$ are different.

Remark 2. The methods set forth in the present note are also applicable to the study of elliptic systems of pseudodifferential operators degenerating on Γ.

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Note: Figure translations are in progress. See original paper for figures.