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Abstract

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CYBERNETICS AND CONTROL THEORY

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ON THE THEORY OF BRANCHING SYSTEMS

Consider a system of ordinary differential equations reduced to normal form:

$$dy/dt = f(y,t), \tag{1}$$

where y, f are n-dimensional vector functions, and the right-hand sides f of system (1) belong to some locally compact set F, which we shall call the space of right-hand sides. We divide the numerical line (the time axis) T by division points $\tau_0, \tau_1, \ldots, \tau_k$ into a finite number of half-intervals $(\tau_0, \tau_1], (\tau_1, \tau_2], \ldots, (\tau_k, \infty)$, to each of which there is assigned some locally compact subset $F_q \subset F$. We agree that, if $f(y,t) \in F_q$, then system (1) has on the interval $(t_q, t_{q+1}) \supset [\tau_q, \tau_{q+1}]$ a unique solution whose integral trajectory passes through a prescribed point $(y(\tau_q), \tau_q) \in Y \times T$, where Y is the phase space of system (1). The topological product $E_q = F_q \times Y$ will be called the state space of the q-th section, and its element $z_q = (f_q, y) \in E_q$ the state of system (1). Thus, the concept of a phase state $y \in Y$ should be distinguished from the concept of a state $z_q = (f_q, y) \in E_q$. The set E_0 will be called the set of initial states.

Let probability measures $P_{E0}(z_0), P_{F1}(f, z_0), P_{F2}(f, z_1), \ldots, \ldots, P_{fk}(f, z_{k-1})$ be given on the Borel sets of the locally compact spaces $E_0, F_1, \ldots, F_k$, respectively, and let the measure $P_{fq}(f, z_{q-1})$ depend, as on a parameter, on the state z_{q-1} of the $(q-1)$-st section of the system.

To each state of the q-th section of the system $z_q^0 = (f_q^0, y^0) \in E_q$ we assign the solution $y(t, z_q^0)$ of system (1) for $f = f_q^0$ and initial conditions $y^0 = y(\tau_q)$, defined on the segment $[\tau_q, \tau_{q+1}]$.

If a probability measure P_{Eq} is given on the system of Borel sets B_q of the space E_q, then the triple set $\{E_q, B_q, P_{Eq}\}$ may be regarded as the basic probability space ([1]). If $\{y(t, z_q^0) : z_q^0 \in E_q\} = y_q^*(t)$ is the set of solutions of system (1) corresponding to all possible points $z_q^0 \in E_q$, then the set $y_q^*(t)$, together with the probability space $\{E_q, B_q, P_{Eq}\}$, is a random process defined on the segment

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$[\tau_q, \tau_{q+1}]$. We shall call this random process the q-th section of the branching process of solutions of system (1).

To each sample function $y(t; z_q^0) = y(t; (f_q^0, y^0))$ we assign the point $z'_q = (f_q^0, y(\tau_{q+1}, z_q^0))$, so that a prescribed continuous mapping of the set E_q into itself is obtained: $z_q^0 \to y(t, z_q^0) \to z'_q$. The probability distribution P_{Eq} on E_q thereby induces a new probability distribution P'_{Eq} on E_q, namely, for any measurable set $E'_{q0} \subset E_q$ we have $P'_{Eq}(E'_{q0}) = P_{Eq}(E_{q0})$, where E_{q0} is the set of all those points $z_q^0 \in E_q$ whose images $z'_q \in E'_{q0}$ under the mapping $z_q^0 \to y(t, z_q^0) \to z'_q$.

To each measurable set $E_{(q+1)0} \subset E_{q+1}$ we assign the measurable set $F_{(q+1)0} \subset F_{q+1}$, equal to the set of all functions f_{q+1} corresponding to points $z_{q+1} \in E_{(q+1)0}$, and the set $E'_{q0} \subset E_q$ of points $z'_q = (f_q, y(\tau_{q+1})) \in E_q$ whose phase states $y(\tau_{q+1})$ coincide with the phase states of the points $z_{q+1} \in E_{(q+1)0}$. We put, by definition,

$$P_{E(q+1)}(E_{(q+1)0}) = \int_{E'_{q0}} \int_{F_{(q+1)0}} dP_{F(q+1)}(f, z_q) dP_{Eq}(z_q). \tag{2}$$

Definition 1. We shall call the system (1) **branching** if the half-intervals $(\tau_0, \tau_1], \dots, (\tau_k, \infty)$, the sets $E_0, F_1, \dots, F_k$, and the probability measures $P_{E0}, P_{F1}, \dots, P_{Fk}$ are given. The probability measures $P_{Eq}(z_q)$, defined on the Borel subsets of $E_q = F_q \times Y$ by relation (2), will be called the probability distribution on the space of initial states of the q-th segment.

By a realization of the branching process of solutions $y_\xi(t)$ we shall mean any continuous real-valued function defined on $[\tau_0, \infty)$ for which there exists a sequence of sample functions $y_{\xi 0}(t), y_{\xi 1}(t), \dots, \dots, y_{\xi k}(t)$ of the solution segments of the branching system (1) such that $t \in (\tau_q, \tau_{q+1}] \Rightarrow y_\xi(t) = y_{\xi q}(t)$.

The asymptotic behavior of branching systems depends on their properties on each branching segment.

Definition 2. A branching system is called **stable** at the point $z_0 \in E_0$ with respect to the attracting set $W \subset Y$ if, for every neighborhood U of the set W, there exist a neighborhood V of the point z_0 and a τ such that, for any realizations $y_{0\xi}(t)$ and $y_\xi(t)$ of the branching process of solutions corresponding respectively to the point z_0 and to an arbitrary point $z \in V$, the difference $(y_\xi(t) - y_{0\xi}(t)) \in U$ for any $t > \tau$.

Definition 3. A branching system is called **stable in the strong sense** at the point $z_{q0} \in E_q$ on the q-th half-interval $(\tau_q, \tau_{q+1}]$ with respect to the attracting set $W_q \subset Y$, for an admissible initial scatter $W_{q0} \subset Y$, if for every neighborhood U of the set W_q there exist a neighborhood U' of the set W_{q0} and a $\tau \in (\tau_q, \tau_{q+1}]$ such that, for any sample function $y(t, z_q)$ for which the difference $[\mathrm{Pr}_Y(z_q) - \mathrm{Pr}_Y(z_{q0})] \in U'$ for all $t \geq \tau$, the difference $[y(t, z_q) - y(t, z_{q0})] \in U$.

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Definition 4. The set E_{00} of all points $z_0 \in E_0$ at which the branching system is stable is called the **stable set**.

In a natural way one introduces the definition of a stable set for each branching segment.

The following theorem holds, establishing the connection between the stability of a branching system and its stability on each branching segment.

Theorem 1. *For stability of a branching system at the point z_0 with respect to the attracting set W, it is sufficient that the following conditions be fulfilled:*

1. For an arbitrary q-th branching segment the stable set E_{q0} is nonempty, so that at each of its points z_{q0} the system is stable in the strong sense on the half-interval $(\tau_q, \tau_{q+1}]$, and the attracting set W_q and the admissible initial scatters W_{q0} of each branching segment are related by the relation $(W_{q-1} - W_{q-1}) \subset W_{q0}$, where $(W_{q-1} - W_{q-1})$ denotes, as usual, the set of differences of all possible pairs of points from W_{q-1}.

2. There exists a realization $y^0_{0\xi}(t)$ of the branching process of solutions corresponding to the initial state z_0 such that the point $(f_q, y) \in E_{q0}$ for any $f_q \in F_q$ and any y for which $[y - y^0_{0\xi}(\tau_q)] \in W_{q-1}$.

3. For the last (k-th) segment, any neighborhood of the set W may be taken as the attracting set.

Proof. a) For an arbitrary q, for every neighborhood $U(W_q)$ of the attracting set W_q, there exists such a neighborhood $U[y^0_{0\xi}(\tau_q)]$ of the point $y^0_{0\xi}(\tau_q) \in Y$, containing all points y for which $[y - y^0_{0\xi}(\tau_q)] \in W_{q-1}$, that for any initial states of the q-th segment $z_{qa} = (f_{qa}, y_a)$ and $z_{qb} = (f_{qb}, y_b)$, for which the phase states $y_a \in$

$\in U[y^0_{0\xi}(\tau_q)]$, $y_b \in U[y^0_{0\xi}(\tau_q)]$, the difference $[y_a(\tau_{q+1}) - y_b(\tau_{q+1})] \in U(W_q)$ for any $f_q \in F_q$.

Indeed, by the conditions of the theorem, from the relation $[y_a - y^0_{0\xi}(\tau_q)] \in W_{q-1}$ it follows that, for any $f_q \in F_q$, the point $z_{qa} = (f_q, y_a) \in E_{q0}$; and since $W_{q0} \supset (W_{q-1} - W_{q-1})$, for every neighborhood $U(W_q)$ of the set W_q there is a neighborhood $U(W_{q0})$ of the set W_{q0} such that, for any $z_{qa} = (f_q, y_a) \in E_{q0}$, the relation $[y_a(\tau_q) - y_b(\tau_q)] \in U(W_{q-1})$ entails $[y_a(\tau_{q+1}) - y_b(\tau_{q+1})] \in U(W_q)$.

On the other hand, if $(y_a - y^0_{0\xi}) \in W_{q-1}$, $(y_b - y^0_{0\xi}) \in W_{q-1}$, then $(y_a - y_b) \in (W_{q-1} - W_{q-1})$; therefore, for any neighborhood $U(W_{q-1} - W_{q-1})$ of the set $(W_{q-1} - W_{q-1})$, there is a neighborhood $U(W_{q-1})$ of the set W_{q-1} such that, if $U(W_{q-1})$ contains both $(y_a - y^0_{0\xi})$ and $(y_b - y^0_{0\xi})$, then $(y_a - y_b) \in U(W_{q-1} - W_{q-1})$.

The constructions given prove assertion a), if one takes into account that the set W_{q0} contains the origin of the coordinates of the phase space, and every neighborhood of the set W_{q0} is a neighborhood of $(W_{q-1} - W_{q-1})$, since, by assumption, $W_{q0} \supset (W_{q-1} - W_{q-1})$.

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For the initial interval, the conditions of item a) are fulfilled by virtue of the continuous dependence of solutions of differential equations on initial conditions and on the form of the right-hand side.

b) Since, by assumption, $(W_{q-1} - W_{q-1}) \subset W_{q0}$, on the basis of item a), for the neighborhood $U(W_{q-1})$ there is a neighborhood $U(W_{q-2})$ such that assertion a) is valid for the neighborhoods $U(W_{q-2})$ and $U(W_{q-1})$, and hence also for the neighborhoods $U(W_{q-2})$ and $U(W_{q0})$. By induction we obtain that, for every neighborhood $U(W_{k-1})$, there is a neighborhood $V(z_0)$ of the point z_0 such that, for $z \in V(z_0)$, the difference $[y_\xi(\tau_k, z) - y_\xi(\tau_k, z_0)] \in U(W_{k0})$.

By virtue of item 3 of the theorem, as the set W_k one may take an open neighborhood of the set W; and since an open set is a neighborhood of each of its points, there are $t_k > \tau_k$ and a neighborhood $U(W_k)$ such that, for any two points $y_a, y_b \in W_{k0}$, the difference $[y_a(t) - y_b(t)] \in W_k$ for any $f_a \in F_k$, $f_b \in F_k$, $t > t_k$. Together with the preceding result, the relation obtained proves the theorem.

This theorem makes it possible to estimate the stability of branching systems, reducing the stability analysis to the successive determination of the set of admissible initial deviations of each system describing the motion on the individual intervals.

Theorem 1 also makes it possible to estimate the probability of stable operation of a system. To do this, we proceed as follows. To each set F_q we assign its measurable subset $F_{q0}(z_0)$, depending, possibly, on the initial state $z_0 \in E_0$ and having the property that, for system (1), sufficient stability conditions are satisfied at the point z_0 if each $F_q \subset F$ is replaced by the set $F_{q0}(z_0) \subset F_q$. In the case where such sets do not exist, put $F_{q0}(z_0) = \emptyset$. Defining probability measures of the sets $F_{q0}(z_0)$ and knowing the probability distribution on the space of initial states, one can estimate the probability of stable operation of the system.

Example. Suppose there is a first-order branching system whose equation is $\dot{y} = f(y,t)$, with two branching intervals corresponding to the half-intervals $(0, \tau_1]$, $(\tau_1, \tau_2]$, and (τ_2, ∞). Let the set F consist of the single element $f_0(y,t) = -ay$, $a > 0$; let the set F_1 consist of constant functions $f(y) = x = \text{const}$, $x \in (-b_1, b_2)$; and let the set F_2 be the set of single-valued continuous functions $f(y) = f^0(y) + \delta$, $\delta \in (-\varepsilon, \varepsilon)$, with $f^0(y) = y$ for $|y| < C$, and $f^0(y) = C \operatorname{sgn} y - y$ for $|y| \geq C$. An arbitrary realization $y_\xi(t)$ of the branching process of solutions, corresponding to $y(0) = y^0$, can be written in the form $y_\xi(t) = y^0 e^{-at}$ for $t \in (0, \tau_1)$, and $y_\xi(t) = y^0 e^{-a\tau_1} + (t - \tau)x$, $x \in (-b_1, b_2)$, for $t \in (\tau_1, \tau_2]$. It is verified directly that on

on the penultimate (first) branching segment the system is strongly stable at every point with respect to the set $W_1 = [-b_1(\tau_2 - \tau_1), b_2(\tau_2 - \tau_1)]$, and on the last (second) segment the system is strongly stable at every point with

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respect to the set $W_2 = [-\varepsilon, \varepsilon]$, with initial phase state $y(\tau_2) \in (-C, +C)$. In this case the set of admissible initial spreads W_{20} can be represented in the form of the interval $W_{20} = (y(\tau_2) - C,\ y(\tau_2) + C)$, and the set $(W_1 - W_1) = [-(b_1 + b_2)(\tau_2 - \tau_1) + (b_1 - b_2)(\tau_2 - \tau_1)]$.

If, as the realization $y_{\xi 0}(t)$ entering into the formulation of the theorem, one chooses the realization corresponding to $y(0) = y^0$, $x = 0$, and $f_2(y) = f^0(y)$, so that $y_\xi(t) = (y^0 e^{-a\tau_1} + C \operatorname{sgn} y^0) e^{t-\tau_2} - C \operatorname{sgn} y^0$ for $t \in (\tau_2, \infty)$, then the sufficient stability condition corresponding to the relation $W_{20} \supset (W_1 - W_1)$ will be the inequality $(b_1 + b_2)(\tau_2 - \tau_1) \leq C - |y^0 e^{-a\tau_1}|$. In the case where the probability distribution on F_2 is given as the uniform distribution on the interval $(-\varepsilon, \varepsilon)$ of the random variable δ, this same inequality is a sufficient condition for the probability of stable operation to be not less than 0.5 for the attracting set $W = [-\varepsilon/2, \varepsilon/2]$.

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Note: Figure translations are in progress. See original paper for figures.

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