# SYSTEMS OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS OF COMPOSITE TYPE WITH ONE DOUBLE FAMILY OF REAL CHARACTERISTICS

MATHEMATICS

1969

SovietRxiv

## Abstract

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UDC 517.944

*MATHEMATICS*

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# SYSTEMS OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS OF COMPOSITE TYPE WITH ONE DOUBLE FAMILY OF REAL CHARACTERISTICS

*(Presented by Academician I. N. Vekua on 29 X 1968)*

In the author' s papers ($^{1-7}$) a theory of boundary value problems was constructed for systems of first-order partial differential equations of composite type with two independent variables.

The present article is devoted to systems of second-order equations of composite type.

1°. Consider, in some bounded simply connected domain $G$ in the plane of the variable $z = x + iy$, the following equation with respect to the complex-valued function $u(z)$:

$$\begin{gathered}D(u) \equiv u_{z\bar{y}} - q(z)u_{zy} + A(z)u_x + B(z)\bar{u}_x + \\ +C(z)u_y + D(z)\bar{u}_y + E(z)u + F(z)\bar{u} = G(z),\end{gathered} \tag{1}$$

where $u_{\bar{z}} = \frac{1}{2}(u_x + iu_y)$, $u_z = \frac{1}{2}(u_x - iu_y)$, and the bar over a function denotes passage to the complex-conjugate value. We shall regard the coefficients and the right-hand side of equation (1) as complex-valued functions, with

$$\begin{gathered}q(z) \in C^2_\nu(\bar{G}), \qquad |q(z)| \le \text{const} < 1, \qquad z \in \bar{G}, \\ A(z), B(z), C(z), D(z) \in C^1_\nu(\bar{G}), \qquad E(z), F(z), G(z) \in C_\nu(\bar{G}).\end{gathered} \tag{2}$$

Equation (1) is equivalent to a system of two real second-order equations with respect to the real and imaginary parts of the function $u(z)$; moreover, this system has a double family of characteristics consisting of the straight lines $x = \text{const}$, and two families of imaginary characteristics, as a result of which it is a system of composite type. In general, if, instead of (1), a general equation of the form

$$A^0(z)u_{xx} + 2B^0(z)u_{xy} + C^0(z)u_{yy} + A(z)u_x + B(z)\bar{u}_x + \\ + C(z)u_y + D(z)\bar{u}_y + E(z)u + F(x)\bar{u} = G(z), \tag{3}$$

is given, possessing the property that the functions $A^0(z)$, $C^0(z)$ are not simultaneously equal to zero in some domain $\widetilde{G}$, then with this equation we associate the fourth-degree polynomial

$$p^0(\lambda) = (A^0(z)\lambda^2 + 2B^0(z)\lambda + C^0(z))(\overline{A^0(z)}\lambda^2 + 2\overline{B^0(z)}\lambda + \overline{C^0(z)})$$

for $A^0(z) \neq 0,\ z \in \widetilde{G}$, or the fourth-degree polynomial

$$p_0(\lambda) = (A^0(z) + 2B^0(z)\lambda + C^0(z)\lambda^2)(\overline{A^0(z)} + 2\overline{B^0(z)}\lambda + \overline{C^0(z)}\lambda^2)$$

for $C^0(z) \neq 0,\ z \in \widetilde{G}$. If the polynomial $p^0(\lambda)$, or the polynomial $p_0(\lambda)$, has simultaneously both real and imaginary roots throughout the domain $\widetilde{G}$, then equation (3) is called an equation of composite type. It is not difficult to convince oneself—

that if $A^0(z) \neq 0$, then the general equation of composite type (1) can always be reduced to the canonical form (1). Indeed, let $\lambda_1(z)$ be a real root of the polynomial $p^0(\lambda)$. Obviously, $\lambda_1(z)$ will be a double root of the polynomial, while the other two roots of the polynomial $p^0(\lambda)$ will be complex-conjugate functions $\lambda_0(z)$ and $\overline{\lambda_0(z)}$, with $\operatorname{Im}\lambda_0(z) \neq 0$, $z \in \overline{G}$. Without loss of generality, we shall assume that

$$\operatorname{Im}\lambda_0(z) \leq \text{const} < 0, \qquad z \in \overline{G}. \tag{4}$$

The real root $\lambda_1(z)$ generates a family of characteristics $\xi(x,y) = \text{const}$, which are integrals of the ordinary differential equation $dy + \lambda_1(z)\,dx = 0$. Let $\eta(x,y)$ be a real function of class $C^2_\nu(\overline{G})$ such that $I = \xi_x\eta_y - \eta_x\xi_y > 0$. Making the change of variables $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ and taking into account that $\xi_x - \lambda_1(z)\xi_y = 0$, it is easy to verify that the principal part of the transformed equation (3) takes the form

$$\widetilde{D}^0(u) \equiv 2\widetilde{B}(\zeta)u_{\xi\eta} + \widetilde{C}(\zeta)u_{\eta\eta}, \qquad \zeta = \xi + i\eta,$$

where $\widetilde{B}(\zeta) = A^0\lambda_1\eta_x + B^0(\lambda_1\eta_y + \eta_x)\xi_y + C^0\xi_y\eta_y$, $\widetilde{C}(\zeta) = A^0\eta_x^2 + 2B^0\eta_x\eta_y + C^0\eta_y^2$, for $p_1(\lambda_1) \equiv 0$, $p_1(\lambda) = A^0\lambda^2 + 2B^0\lambda + C^0$. But since $p_1(\lambda_0) = 0$ and, consequently, $C^0 = -A^0\lambda_0^2 - 2B^0\lambda_0$, we have $\widetilde{C} = [A^0(\eta_x + \lambda_0\eta_y) + 2B^0\eta_y] \times (\eta_x - \lambda_0\eta_y)$,

$$\begin{aligned}2\widetilde{B} &= 2\{A^0(\lambda_1\eta_x - \lambda_0^2\eta_y) + 2B^0[(\lambda_1 - \lambda_0)\eta_y + (\eta_x - \lambda_0\eta_y)]\}\xi_y \\ &= \{(\lambda_1 - \lambda_0)[A^0(\eta_x + \lambda_0\eta_y) + 2B^0\eta_y] + [(\lambda_1 + \lambda_0)A^0 + 2B^0] \times (\eta_x - \lambda_0\eta_y)\}\xi_y \\ &= (\lambda_1 - \lambda_0)[A^0(\eta_x + \lambda_0\eta_y) + 2B^0\eta_y]\xi_y.\end{aligned}$$

Therefore $\widetilde{D}^0(u) \equiv (\lambda_1 - \lambda_0)\xi_y k(\zeta)\{u_{\xi\eta} - \lambda_*(\zeta)u_{\eta\eta}\}$, where the function $k(\zeta) = A^0(\eta_x + \lambda_0\eta_y) + 2B^0\eta_y$ is nonzero by virtue of the condition $I > 0$, and the function $\lambda_*(\zeta) = (\lambda_0\eta_y - \eta_x)/(\lambda_1 - \lambda_0)\xi_y$ is such that $\operatorname{Im}\lambda_*(\zeta) = \operatorname{Im}\lambda_0 \cdot I/|\lambda_1 - \lambda_0|^2 \cdot \xi_y^2 < 0$. Consequently,

$$\widetilde{D}^0(u) \equiv (\lambda_1 - \lambda_0)(1 + i\lambda_*)k\{u_{\overline{\zeta}\eta} - q(\zeta)u_{\zeta\eta}\},$$

where $q(\zeta) = (\lambda_* + i)/(\lambda_* - i)$. Dividing now equation (3), transformed to the variables $\xi, \eta$, by the nonzero function $(\lambda_1 - \lambda_0)(1 + i\lambda_*)k$ and taking into account that

$$|q(\zeta)| = |(\lambda_* + i)/(\lambda_* - i)| \leq \text{const} < 1,$$

we obtain an equation of the form (1). If $C^0(z) \neq 0$, $z \in \overline{G}$, then equation (3) is reduced in an analogous way to a canonical form with principal part $u_{zx} - q_{\overline{z}x}$.

2°. By a regular solution of equation (1) we shall henceforth understand twice continuously differentiable solutions. We now construct an integral equation of Fredholm type equivalent to equation (1). To this end we write equation (1) in the form

$$u_{zy} - q(z)u_{\overline{z}y} + C(z)u_y + D(z)\overline{u}_y = G_1(z), \tag{5}$$

where

$$G_1(z) = G(z) - A(z)u_x - B(z)\overline{u}_x - E(z)u - F(z)\overline{u}. \tag{6}$$

Let $Z(\zeta, z)$ be the fundamental solution of the Beltrami equation (see ([2,4])): $u_{\overline{z}} - q(z)u_z = 0$, and let $u_0(z)$ be the general solution of the homogeneous equation (5). Then the general solution of equation (5) has the form:

$$u(z) = u_0(z) + \iint_G \{K_1^0(\zeta, z)G_1(\zeta) + K_2^0(\zeta, z)\overline{G_1(\zeta)}\}\, dG_\zeta, \tag{7}$$

where

$$K_j^0(\zeta,z)=\int_{\sigma_2(x)}^{y}K_j(\zeta,x+i\sigma)\,d\sigma;$$

$$K_1(\zeta,z)=-\frac{1}{\pi}\left(Z(\zeta,z)+\iint_G\Gamma_1(t,z)Z(\zeta,t)\,dG_t\right);$$

$$K_2(\zeta,z)=-\frac{1}{\pi}\iint_G\Gamma_2(t,z)\overline{Z(\zeta,t)}\,dG_t;$$

$y=\sigma_j(x)$ is the equation of the curve $\gamma_j\subset\Gamma$; $\Gamma_1(\xi,z)$, $\Gamma_2(\xi,z)$ satisfy the following Fredholm system of integral equations:

$$\Gamma_1(\xi,z)-\frac{1}{\pi}\iint_G Z(t,z)\{C(t)\Gamma_1(\xi,t)+D(t)\overline{\Gamma_2(\xi,t)}\}\,dG_t=\frac{C(\xi)}{\pi}Z(\xi,z),$$

$$\Gamma_2(\xi,z)-\frac{1}{\pi}\iint_G Z(t,z)\{D(t)\overline{\Gamma_1(\xi,z)}+C(t)\Gamma_2(\xi,t)\}\,dG_t=\frac{D(\xi)}{\pi}Z(\xi,z). \quad (8)$$

Taking into account equality (6), in accordance with (7) we are convinced that all regular solutions of (1) satisfy the integral equation

$$u(z)-\iint_G\{P(\xi,z)u(\xi)+Q(\xi,z)\overline{u(\xi)}\}\,dG_\xi=u^0(z), \quad (9)$$

where

$$P(\xi,z)=\frac{\partial}{\partial\xi}(A(\xi)K_1^0(\xi,z)+\overline{B(\xi)}K_2^0(\xi,z))-E(\xi)K_1^0(\xi,z)-\overline{F(\xi)}K_2^0(\xi,z),$$

$$Q(\xi,z)=\frac{\partial}{\partial\xi}(B(\xi)K_1^0(\xi,z)+\overline{A(\xi)}K_2^0(\xi,z))-F(\xi)K_1^0(\xi,z)-\overline{E(\xi)}K_2^0(\xi,z),$$

$$\begin{aligned}u^0(z)=u_0(z)+\int_\Gamma&\{[A(\xi)K_1^0(\xi,z)+\overline{B(\xi)}K_2^0(\xi,z)]u(\xi)\\&+[B(\xi)K_1^0(\xi,z)+\overline{A(\xi)}K_0^2(\xi,z)]\overline{u(\xi)}\}\,d\eta\\+\iint_G&\{K_1^0(\xi,z)G(\xi)+K_2^0(\xi,z)\overline{G(\xi)}\}\,dG_\xi.\end{aligned}$$

It is easy to verify that the function $u^0(z)$ is the general solution of the equation

$$u_{\overline{z}y} - q(z)u_{zy} + C(z)u_y + D(x)\overline{u}_y = G(z). \tag{10}$$

Conversely, assuming that $u(z)$ is a solution of the integral equation (9) with a right-hand side that is a solution of equation (10), it is easy to show that $u(z)$ will be a solution of equation (1). Consequently, equations (1) and (9) are equivalent.

3°. The kernels $P(\xi, z)$, $Q(\xi, z)$, by virtue of their construction, are continuous everywhere except for the point $\xi = z$, in a neighborhood of which they satisfy the inequalities

$$|P(\xi, z)|,\ |Q(\xi, z)| \leq \text{const} \cdot |\xi - z|^{-1}.$$

Therefore equation (9) is Fredholm. The general solution of equation (9) has the form

$$u(z) = u^0(z) + \iint_G \{R_1(\xi, z)u^0(\xi) + R_2(\xi, z)\overline{u^0(\xi)}\}\, dG_\xi + \sum_{k=1}^{N} C_k u_k(z), \tag{11}$$

where $R_j(\xi, z)$ are the generalized resolvents of equation (9), which, as is not difficult to see, satisfy the Fredholm system

$$R_1(\xi, z) - \iint_G \{P(t, z)R_1(\xi, t) + Q(t, z)\overline{R_2(\xi, t)}\}\, dG_t = P(\xi, z),$$

$$R_2(\xi, z) - \iint_G \{P(t, z)R_2(\xi, t) + Q(t, z)\overline{R_1(\xi, t)}\}\, dG_t = G(\xi, z); \tag{12}$$

$C_k$ are arbitrary real constants; $u_k(z)$ are linearly independent solutions of the homogeneous equation corresponding to (9), and $u^0(z)$ is a solution of equation (10) satisfying conditions of the form

$$\iint_G u^0(z) \times u_k^*(z)\, dG_z = 0, \quad k = 1, 2, \dots, N.$$

It follows from formula (11) that, in order to have a complete representation of the structure of the solutions of equation (1), it suffices to know the structure of the general solution of equation (10). But the general solution of the latter equation has the form

$$u^0(z) = u_0(z) + \iint_G \{K_1^0(\xi, z)G(\xi) + K_2^0(\xi, z)\overline{G(\xi)}\}\, dG_\xi,$$

where $u_0(z)$ is the general solution of the homogeneous equation (10) ($G(z) \equiv 0$). If $\varphi(z)$ is the general solution of the uniformly elliptic equation $\varphi_z - q(z)\varphi_z + G(z)\varphi + D(z)\bar{\varphi} = 0$, then the function $u_0(z)$, obviously, has the form

$$u_0(z) = \omega(x) + \int_{\sigma_2(x)}^{y} \varphi(x + i\sigma)\, d\sigma,$$

where $\omega(x)$ is an arbitrary complex-valued function of the variable $x$. It follows from this that the general solution of equation (1) is completely determined by formula (11) in terms of an arbitrary function of one variable $\omega(x)$ and an arbitrary solution $\varphi(z)$ of a uniformly elliptic equation satisfying a finite number of additional conditions, and also in terms of a finite number of real constants. Formula (11) will be used essentially in our subsequent work for the study of boundary-value problems for equation (1).

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Received
28 X 1968

## REFERENCES

1. A. Dzhuraev, *Dokl. AN TadzhSSR*, **7**, No. 10 (1964).
2. A. Dzhuraev, *DAN*, **171**, No. 2 (1966).
3. A. Dzhuraev, *DAN*, **173**, No. 2 (1967).
4. A. Dzhuraev, *Izv. AN SSSR, Ser. Mat.*, **31**, No. 3 (1967).
5. A. Dzhuraev, *Sibirsk. Mat. Zhurn.*, **7**, No. 2 (1967).
6. A. Dzhuraev, *DAN*, **181**, No. 3 (1968).
7. A. Dzhuraev, *DAN*, **181**, No. 4 (1968).
8. I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959.

*Note: Figure translations are in progress. See original paper for figures.*

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