



---

Soviet-era science, translated into English

# ON A PROPERTY OF LOGARITHMICALLY CONCAVE FUNCTIONS

MATHEMATICS

1969

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.68812>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 519.21

*MATHEMATICS*

Yu. S. DAVYDOVICH, B. I. KORENBLYUM, B. I. KHATS T

## ON A PROPERTY OF LOGARITHMICALLY CONCAVE FUNCTIONS

*(Presented by Academician N. N. Bogolyubov on 19 IX 1968)*

1°. The main result of the present note is a theorem establishing that the convolution of two logarithmically concave functions of  $n$  variables is a logarithmically concave function.\* We also give some particular cases of this theorem which, in our opinion, are of independent interest.

2°. **Definition 1.** A nonnegative function  $f(x)$  ( $x = (x_1, \dots, x_n) \in R^n$ ) is called **logarithmically concave** if:

a) the set  $E_0^f = E\{x \in R^n : f(x) > 0\}$  is open; b)  $f(x)$  is continuous on  $E_0^f$ ; c) for any  $x', x'' \in R^n$

$$f[(x' + x'')/2] \geq \sqrt{f(x')f(x'')}. \quad (1)$$

**Definition 2.** We shall denote by  $\mathfrak{C}$  the class of logarithmically concave functions in  $R^n$ ; by  $\mathfrak{C}_1$  the subclass of functions  $f(x)$  such that

$$\lim_{|x| \rightarrow \infty} \frac{f(x)}{e^{\varepsilon|x|}} = 0$$

for any  $\varepsilon > 0$

$$\left( |x| = \left( \sum_{i=1}^n x_i^2 \right)^{1/2} \right),$$

and by  $\mathfrak{C}_0$  the subclass of summable functions.

The following properties follow easily from the definitions:

- 1) If  $f(x) \in \mathfrak{C}$ , then the sets  $E_\tau^f = E\{x \in R^n : f(x) > \tau\}$  ( $\tau \geq 0$ ) are convex.
- 2) If  $f(x) \in \mathfrak{C}$  and  $g(x) \in \mathfrak{C}$ , then  $f(x)g(x) \in \mathfrak{C}$ .
- 3) Let  $E_0 \subset R^n$  be a convex open set, and let  $\tilde{f}(x) \geq 0$  be defined on  $E_0$  and satisfy inequality (1) for any  $x', x'' \in E_0$ . Then the function

$$f(x) = \begin{cases} \tilde{f}(x), & (x \in E_0), \\ 0, & (x \in R^n \setminus E_0) \end{cases}$$

is logarithmically concave.

- 4) In order that the characteristic function  $\chi_{E_0}(x)$  of some set  $E_0 \subset R^n$  belong to  $\mathfrak{C}$ , it is necessary and sufficient that  $E_0$  be open and convex.
- 5) If  $f(x) \in \mathfrak{C}$ , then for any  $x', x'' \in E_0^f$  and  $\lambda$  ( $0 \leq \lambda \leq 1$ ) the inequality

$$f[\lambda x' + (1 - \lambda)x''] \geq [f(x')]^\lambda [f(x'')]^{1-\lambda} \quad (2)$$

holds.

- 6) If  $f(x) \in \mathfrak{C}$ , then  $f(x)$  is bounded in every ball of finite radius.
- 7) Let  $f(x) \in \mathfrak{C}$ . The following assertions are equivalent:
  - a)  $f(x) \in \mathfrak{C}_0$ ;
  - b) there exist positive numbers  $C$  and  $a$  such that  $f(x) \leq Ce^{-a|x|}$ ;
  - c)

$$\lim_{|x| \rightarrow \infty} f(x) = 0.$$

We shall prove the last property.

\* For  $n = 1$  this theorem, in its essential part, is contained in a result of I. A. Ibragimov <sup>(1)</sup>.

I. Let us first show that a) implies b). Suppose the contrary: let  $f(x) \in \mathfrak{C}_0$ , but for any  $C > 0$ ,  $\alpha > 0$  there exists an  $x \in R^n$  such that  $f(x) > Ce^{-\alpha|x|}$ . In this case one may assume that  $|x| > N$ , where  $N$  is an arbitrary positive number. Without loss of generality one may also assume  $f(0) = 1$ . Put  $C = 1$ ,  $\alpha = 1/k$ , and let  $x_k \in R^n$  be such that

$$f(x_k) > e^{-|x_k|/k}, \quad |x_k| > k. \quad (3)$$

By compactness of the unit sphere  $S \subset R^n$ , from the sequence  $\xi_k = x_k/|x_k|$  one can extract a convergent subsequence (which we shall also denote by  $\xi_k$ ):  $\xi_k \rightarrow \xi_0 \in S$ . Consider the ray  $l_0 : y = \xi_0 t$  ( $t > 0$ ) and show that at every point of it  $f(y) \geq 1$ . Let  $\eta_k = \frac{x_k}{|x_k|} t$ . Obviously,

$$\lim_{k \rightarrow \infty} \eta_k = y \in l_0.$$

For sufficiently large  $k$ ,  $\eta_k = (1 - \lambda) \cdot 0 + \lambda x_k$ , where  $\lambda = t/|x_k| < 1$ , and by virtue of (2) and (3)

$$f(\eta_k) \geq [f(0)]^{1-\lambda} [f(x_k)]^\lambda > e^{-\frac{\lambda}{k}|x_k|} = e^{-t/k}.$$

Hence\*  $f(y) = \lim_{k \rightarrow \infty} f(\eta_k) \geq 1$ . It is now easy to see that  $f(x)$  is not summable. Let  $R_0^{n-1}$  be the subspace orthogonal to the ray  $l_0$ , and let  $\sigma \subset R_0^{n-1}$  be an  $(n-1)$ -dimensional closed ball of radius  $r$  such that  $f(u) \geq 1/2$ ,  $u \in \sigma$ . Obviously, the set  $Z$  of points of the form  $z = \xi_0 t + u$  ( $t > 0$ ,  $u \in \sigma$ ) forms a cylindrical body of radius  $r$  with axis  $l_0$ . It remains to show that  $f(z) \geq 1/2$  ( $z \in Z$ ). On  $l_0$ ,  $f(z) \geq 1$ . If  $z$  is some interior point of  $Z$ ,  $z \notin l_0$ , then the two-dimensional plane passing through  $z$  and  $l_0$  intersects the boundary of the ball  $\sigma$  in two points; one of them, say  $\zeta$ , lies with  $z$  on one side of the axis  $l_0$ . On  $l_0$  there is a point  $y$  such that  $z$  lies on the segment joining  $\zeta$  and  $y$ . By virtue of (2),  $f(z) \geq 1/2$ .

II. That a) and c) follow from b) is obvious. Let us show that b) follows from c). Indeed, if  $f(x) \in \mathfrak{C}$  does not satisfy b), then, as shown in I, there exists a ray along which  $f(x) \geq 1$ , i.e., c) cannot hold.

3°. **Theorem.**  $\mathfrak{C}_0 * \mathfrak{C}_1 \subset \mathfrak{C}_1$ ;  $\mathfrak{C}_0 * \mathfrak{C}_0 \subset \mathfrak{C}_0$ .

It is enough to prove the latter inclusion. We first establish two lemmas.

**Lemma 1.** Let  $A, B$  be open convex bounded centrally symmetric sets in  $R^n$ ;  $A + x$  the parallel translate of  $A$  by the vector  $x$ ; and  $\mu(x)$  the  $n$ -dimensional volume of the set  $(A + x) \cap B$ . Then

$$\mu(0) \geq \mu(x) \quad (x \in R^n). \quad (4)$$

**Proof.** Let  $C = (A + x) \cap B$ . Consider the sets  $D = C - x$ ,  $C' = -C$ ,  $D' = -D$ . Obviously,  $C \subset B$ ,  $C' \subset B$ ,  $D \subset A$ ,  $D' \subset A$ ,  $D' = C' + x$ . Now construct the set  $K = \frac{1}{2}(C + C') = \frac{1}{2}(D + D')$ . Since  $K \subset A$ ,  $K \subset B$ , we have  $K \subset A \cap B$ , and  $V_K \leq \mu(0)$  ( $V_Q$  denotes the volume of the set  $Q \subset R^n$ ). On the other hand, from the Brunn-Minkowski inequality (2) it follows that

$$\sqrt[n]{V_K} \geq \frac{1}{2} \sqrt[n]{V_C} + \frac{1}{2} \sqrt[n]{V_{C'}}.$$

Since  $V_C = V_{C'}$ , we obtain  $V_K \geq V_C = \mu(x)$ . The lemma is proved.

**Lemma 2.** Let  $F(x), G(x)$  be nonnegative functions on  $R^n$  possessing the following properties:

- a)  $F(-x) = F(x)$ ,  $G(-x) = G(x)$ ;  
 b) the sets  $E_\tau^F = E\{x \in R^n : F(x) > \tau\}$ ,  $E_\tau^G = E\{x \in R^n : G(x) > \tau\}$  ( $\tau \geq 0$ ) are convex and open.

---

\* The continuity of  $f(x)$  on  $l_0$  follows from the fact that, as is easy to show,  $l_0 \subset E_0$ .

Then for arbitrary  $p \in R^n$

$$\int_{R^n} F(x)G(x) dx \geq \int_{R^n} F(x-p)G(x) dx. \quad (5)$$

Inequality (5) should be understood in the sense that convergence of the integral on the left entails convergence of the integral on the right, while divergence of the latter entails divergence of the integral on the left.

**Proof.** Denote the characteristic functions of the sets  $E_\tau^F$  and  $E_\tau^G$  by  $\varphi_\tau(x)$  and  $\psi_\tau(x)$ , respectively. It is clear that the functions  $F(x)$  and  $G(x)$  can be represented in the form

$$F(x) = \int_0^\infty \varphi_\tau(x) d\tau; \quad G(x) = \int_0^\infty \psi_\tau(x) d\tau \quad (x \in R^n).$$

Taking into account that the sets  $E_\tau^F$  and  $E_\tau^G$  are centrally symmetric, and applying Lemma 1, we find

$$\begin{aligned} \int_{R^n} F(x-p)G(x) dx &= \int_0^\infty \int_0^\infty \left[ \int_{R^n} \varphi_{\tau_1}(x-p)\psi_{\tau_2}(x) dx \right] d\tau_1 d\tau_2 \leq \\ &\leq \int_0^\infty \int_0^\infty \left[ \int_{R^n} \varphi_{\tau_1}(x)\psi_{\tau_2}(x) dx \right] d\tau_1 d\tau_2 = \int_{R^n} F(x)G(x) dx. \end{aligned}$$

We now pass to the proof of the theorem. Let  $f(x), g(x) \in \mathfrak{C}_0$ . We shall show that the function

$$h(x) = (f * g)(x) = \int_{R^n} f(x-s)g(s) ds \quad (ds = ds_1 \dots ds_n)$$

is logarithmically concave. Let  $x, y \in R^n$ . Clearly,

$$\left[ h\left(\frac{x+y}{2}\right) \right]^2 = \int_{R^n} \int_{R^n} f\left(\frac{x+y}{2}-s\right) f\left(\frac{x+y}{2}-t\right) g(s)g(t) ds dt,$$

$$h(x)h(y) = \int_{R^n} \int_{R^n} f(x-s)f(y-t)g(s)g(t) ds dt.$$

In these  $2n$ -dimensional integrals make the change of variables:

$$s = \frac{1}{2}(u-v), \quad t = \frac{1}{2}(u+v), \quad ds dt = 2^{-n} du dv.$$

We obtain

$$2^n \left[ h\left(\frac{x+y}{2}\right) \right]^2 = \int_{R^n} \int_{R^n} F(u,v)G(u,v) du dv,$$

$$2^n h(x)h(y) = \int_{R^n} \int_{R^n} F(u,v-x+y)G(u,v) du dv,$$

where

$$F(u,v) = f\left[\frac{x+y-u-v}{2}\right] f\left[\frac{x+y-u+v}{2}\right]; \quad G(u,v) = g\left[\frac{u+v}{2}\right] g\left[\frac{u-v}{2}\right].$$

Since the functions  $F(u,v)$ ,  $G(u,v)$  are logarithmically concave and even in the argument  $v$ , applying Lemma 2, we find

$$\int_{R^n} F(u,v)G(u,v) dv \geq \int_{R^n} F(u,v-x+y)G(u,v) dv \quad (u \in R^n).$$

Integrating with respect to  $u$ , we obtain

$$\left[ h\left(\frac{x+y}{2}\right) \right]^2 \geq h(x)h(y).$$

The theorem is proved.

§ 4. We note some special cases of the theorem proved.

1) The gamma-distribution function

$$F_1(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-\beta t} dt \quad (x \geq 0); \quad F_1(x) = 0 \quad (x < 0),$$

is logarithmically concave for  $\alpha \geq 1$ ,  $\beta > 0$ .

2) The function of the multidimensional normal distribution

$$F_2(x_1, \dots, x_n) = \frac{\sqrt{|A|}}{(2\pi)^{n/2}} \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^n a_{ij}(t_i - b_i)(t_j - b_j) \right\} dt_1 \dots dt_n,$$

where  $A = (a_{ij})$  is a positive definite matrix, and also the functions

$$\Phi_2(x_1, \dots, x_n) = \frac{\sqrt{|A|}}{(2\pi)^{n/2}} \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_n} (x_1 - t_1)^{k_1} \dots (x_n - t_n)^{k_n} \times \exp \left\{ -\frac{1}{2} \sum_{i,j=1}^n a_{ij}(t_i - b_i)(t_j - b_j) \right\} dt_1 \dots dt_n \quad (k_i \geq 0)$$

are logarithmically concave.

3) The function

$$F_3(x) = \int_{K+x} \exp \left\{ -\sum_{i,j=1}^n a_{ij}t_it_j \right\} dt_1 \dots dt_n \quad (x \in R^n),$$

where  $K$  is an arbitrary convex set in  $R^n$ , is logarithmically concave.

4) Repeated convolutions of characteristic functions of convex sets in  $R^n$  and their limits are logarithmically concave functions. For example, the Mandelbrot-Van Ness functions <sup>(3)</sup>, whose Fourier transform has the form

$$\prod_{n=1}^{\infty} \frac{\sin \mu_n \xi}{\mu_n \xi} \quad \left( \sum_{n=1}^{\infty} \mu_n < \infty \right),$$

are logarithmically concave.

5°. As is known <sup>(4)</sup>, the convolution of two even unimodal functions in  $R^1$  is an even unimodal function. In the multidimensional case it is natural to understand by a unimodal function a function  $f(x)$ ,  $x \in R^n$ , for which the sets  $E_\tau^f = E\{x \in R^n : f(x) > \tau\}$  are convex ( $\tau \geq 0$ ). However, in  $R^n$  ( $n \geq 2$ ) the convolution of even unimodal functions will not in general be a unimodal function.

**Example.** Let  $\chi_A(x)$ ,  $\chi_B(x)$  be the characteristic functions of the sets

$$A = \{x = (x_1, x_2) : -1 \leq x_1 \leq 1; -b \leq x_2 \leq b\}$$

and

$$B = \{x = (x_1, x_2) : -1 \leq x_1 \leq 1; x_1 - \delta \leq x_2 \leq x_1 + \delta\}.$$

Consider the even unimodal function

$$\varphi(x) = \chi_A(x) + N\chi_B(x),$$

where  $0 < \delta < b < N$ . For sufficiently small  $\delta$  and sufficiently large  $N$ , the convolution  $(\varphi * \varphi)(x)$  is not unimodal. To see this, it is enough to note that  $(\varphi * \varphi)(x)$  along the line  $x_2 = 1$ ,  $-\infty < x_2 < \infty$ , has at least two local maxima (at  $x_2 = 0$  and  $x_2 = 1$ ).

Scientific Research Institute  
of Automated Planning and Management Systems  
in Construction of Gosstroy of the Ukrainian SSR

Received  
17 IX 1968

### CITED LITERATURE

1. I. A. Ibragimov, *Probability Theory and Its Applications*, **1**, no. 2 (1956).
2. B. N. Delone, *Uspekhi Matematicheskikh Nauk*, **2**, 39 (1936).
3. S. Mandelbrojt, *Adjoining Series. Regularization of Sequences. Applications*, Moscow, 1955.
4. I. A. Ibragimov, Yu. V. Linnik, *Independent and Stationarily Related Random Variables*, Moscow, 1965.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*