LOCAL PROPERTIES OF THE SOLUTION OF THE TRANSPORT EQUATION

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LOCAL PROPERTIES OF THE SOLUTION OF THE TRANSPORT EQUATION

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For a correct understanding of the peculiarities of the physical phenomena accompanying the process of propagation of radiation in matter, and for the successful development of numerical methods for solving the transport equation in problems with complicated geometry, it becomes necessary to investigate the local properties of this solution: obtaining the sharpest possible estimates, analyzing regions of continuity and differentiability, studying behavior in neighborhoods of boundaries and discontinuity surfaces, etc. The behavior of the solution is so complicated ([1]) that the theory of boundary-value problems for the transport equation is constructed, as a rule ([1−3]), on the basis of the introduction of generalized solutions. A uniform estimate of the solution itself in one-speed problems is given by the maximum principle ([4]). The first smoothness estimates were obtained by V. S. Vladimirov in the study of a one-speed problem with isotropic scattering in a spherically symmetric domain ([5], [6]).

In the present work we set forth some results of an investigation of the smoothness and differentiability properties and of the peculiarities of the solution of an inhomogeneous boundary-value problem for the one-speed transport equation in a bounded open set (domain) G of the Euclidean space R_3. A detailed exposition is contained in ([7]).

We shall assume that the domain G decomposes into a finite number J of subdomains G_j, each of which is an open set with a piecewise smooth boundary surface γ_j; and that any straight line Π_{rs} passing through a point $\mathbf{r} \in G$ in the direction $\mathbf{s}$ intersects the surface $\gamma = \bigcup_1^J \gamma_j$ only in a finite number of points

$$\mathbf{r}^n = \mathbf{r} + \xi_n \mathbf{s}, \qquad n = 1, 2, \dots, \mathfrak{N}(\mathbf{r}, \mathbf{s}), \qquad \sup_{G\times\Omega} \mathfrak{N}(\mathbf{r}, \mathbf{s}) \le \mathfrak{N}_0 < \infty,$$

$$\xi_1 < 0, \qquad \xi_{2m-1} < \xi_{2m} \le \xi_{2m+1} < d, \qquad d = \operatorname{diam} G.$$

The boundary-value problem under consideration for the function $\Phi(\mathbf{r}, \mathbf{s})$—the intensity of radiation at the point $\mathbf{r}$ in the direction $\mathbf{s}$—is formulated as follows:

$$\mathbf{s}\nabla\Phi + \sigma(\mathbf{r})\Phi(\mathbf{r},\mathbf{s}) = \int_{\Omega} g(\mathbf{r},\mathbf{ss}')\Phi(\mathbf{r},\mathbf{s}')\,d\mathbf{s}' + f(\mathbf{r},\mathbf{s}), \qquad \mathbf{r}\in G,\ \mathbf{s}\in\Omega; \quad (1)$$

$$\Phi(\mathbf{r}+\xi_0\mathbf{s},\mathbf{s}) = 0, \qquad \Phi(\mathbf{r}+\xi_{2m+1}\mathbf{s},\mathbf{s}) = \varphi(\mathbf{r}+\xi_{2m+1}\mathbf{s},\mathbf{s}) + \Phi(\mathbf{r}+\xi_{2m}\mathbf{s},\mathbf{s}),$$

$$m = 1, 2, \ldots, \mathfrak{N}/2.$$

Here Ω is the unit sphere; the coefficients $\sigma(\mathbf{r})$, $g(\mathbf{r},\mathbf{ss}')$, and $f(\mathbf{r},\mathbf{s})$ are assumed to be uniquely defined in G, $G\times[-1,+1]$, and $G\times\Omega$, respectively, while the function $\varphi(\mathbf{r},\mathbf{s})$ is uniquely defined at all points $\mathbf{r}_\gamma = \mathbf{r}+\xi_n\mathbf{s}\in\gamma$ for directions $\mathbf{s}\in\Omega_-$, corresponding to external radiation entering G at this point. We shall assume that outside G, $\sigma = g = f = 0$.

Under the conditions

$$0 < \sigma_0 \le \sigma(\mathbf{r}) \le \sigma_1 < \infty \quad \text{in } G, \qquad (2)$$

$$g(\mathbf{r},\mathbf{ss}') = \sum_{i=1}^{I} a_i(\mathbf{r}) g_i(\mathbf{ss}'), \qquad \sup_G a_i \le A < \infty, \qquad g_i \in L_1(-1,+1),$$

I is finite, $f\in L_\infty(G\times\Omega)$, $\varphi\in L_\infty(\gamma\times\Omega_-)$; there exists a generalized solution of this problem which, together with $s\nabla\Phi$, belongs to the space $L_\infty(G\times\Omega)$ with norm

$$\|\Phi\|_\infty = \sup_{G\times\Omega} \operatorname{vrai} \Phi(r,s)$$

and, almost everywhere in $G\times\Omega$,

$$\Phi(r,s) = \int_{\xi_1}^{0} \exp\left[-\int_{\xi}^{0} \sigma(r+\xi' s)\,d\xi'\right] B(r+\xi s, s)\,d\xi + F(r,s), \qquad (3)$$

where

$$B(r,s) = \int_\Omega g(r,ss')\Phi(r,s')\,ds' + f(r,s),$$

$$F(r,s) = \sum_{\xi_{2m+1}<0} \varphi(r+\xi_{2m+1}s, s) \exp\left[-\int_{\xi_{2m+1}}^{0} \sigma(r+\xi' s)\,d\xi'\right]. \qquad (4)$$

Solvability conditions for the corresponding homogeneous problem (1) have also been found.

Denote by $C_{l\alpha}(\bar{Y})$ the space of functions $u(y)$, l times continuously differentiable in the closed domain $\bar{Y} \subset R_n$, with finite magnitude (norm) $(1, 7)$

$$\sum_{k=0}^{l} \sum_{(k)} \max |D^k u| + \sum_{(l)} |D^l u|_{\alpha \bar{Y}} = \|u\|_{l\alpha \bar{Y}},$$

where $D^k u$ is the derivative of u of order k;

$$|u|_{\alpha \bar{Y}} = \sup_{\substack{y_0 \in \bar{Y} \\ 0 \leq \rho \leq \rho_0}} \frac{\operatorname{osc}\{u, K_\rho(y_0)\}}{\tau(\rho)};$$

$\tau(\rho) = \rho^\alpha$ for $0 \leq \alpha < 1$; $\tau(\rho) = \omega(\rho) = \rho(1 + |\ln|\rho||)$ for $\alpha = 1$; $K_\rho(y_0)$ is the ball of radius ρ in Y with center at the point $y_0 \in \bar{Y}$.

Let $L_p^{l\alpha}$ be the set of functions $u(y)$, l times generalized differentiable in Y, for which the inequality

$$\left[\int_Y dy \sum_{k=0}^{l} \sum_{(k)} |D^k(y + \Delta'(y)) - D^k u(y)|^p \right]^{1/p} \leq c_u \Delta^\alpha, \qquad \Delta \equiv \sup_Y \operatorname{vrai} \Delta'(y).$$

Lemma 1. Let $f(s) \in L_q(\Omega)$. Then, for $p^{-1} = 1 - q^{-1}$ and $g(\mu) \in L_p^{l\alpha}(-1, +1)$, $l = 0, 1, ...$, $\alpha \in [0, 1]$, there exists any derivative $D^k \hat{g} f$ $(k = 0, 1, \ldots, l)$ of the function

$$\hat{g} f(s) = \int_\Omega g(ss') f(s') \, ds',$$

and moreover $\hat{g} f \in C_{l\alpha}(\Omega)$.

Consequently, under conditions (2), $N \equiv \hat{g}\Phi \in C_{l\alpha}(\Omega)$ at every point $r \in G$, where $g(r, ss') \in L_1^{l\alpha}(-1, +1)$, since in this case $\Phi \in L_\infty(G \times \Omega)$ and $p = 1$.

From the definition of the source function N it follows that its degree of smoothness with respect to the spatial variables r is at least the same as that of the functions g and Φ. However, owing to the absolute continuity of $\Phi(r, s)$ along the rays Π_{rs}, N turns out to be a smoother function than Φ.

Theorem 1. Let the γ_j—the boundaries of the subdomains G_j $(j = 1, 2, \ldots, J)$ —be piecewise smooth surfaces of class $C_{l\alpha}$ $(l \geq 1)$, $\varphi(r_\gamma, s) = 0$, and let $\bar{B}$, g, and σ satisfy the conditions: 1) $\sigma \geq 0$, $\sigma \in C_{0\alpha}(\bar{G_j})$; 2) $B \in C_{0\alpha}(\Omega)$ in G; 3)

$g(r, ss') \in L_\infty(G \times (-1, +1))$, $a_i \in C_{0\alpha}(K_R(r))$, $g_i \in L_1^{0\alpha}(-1, +1)$, $\alpha > 0$, I finite.

Then $N \in C_{0\alpha}(\Omega \times K_R(r))$, where $K_R(r)$ is the open ball of radius R with center at the point r, contained in G.

The proof is based on representing $N(r, s)$ by the integral

$$N(r, s) = \int_G dr'\, g(r, ss') K(r, r') B(r', s'), \tag{5}$$

where $K(\mathbf{r}, \mathbf{r}') = |\mathbf{r} - \mathbf{r}'|^{-2} \exp\{-w_\sigma(\mathbf{r}, \mathbf{r}')\}$,

$$w_\sigma(\mathbf{r}, \mathbf{r}') = |\mathbf{r} - \mathbf{r}'| \int_0^1 \sigma(\mathbf{r}t + (1-t)\mathbf{r}')\, dt, \qquad \mathbf{s}' = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

The difficulty of the proof, just as in the problem with isotropic scattering ($g_i \equiv 1$) ([1]), is explained by the dependence of the function playing the role of the potential density not only on the spatial and angular coordinates, but also on the functional w_σ.

Theorem 2. *Suppose that, for some $\alpha > 0$, the following conditions are satisfied:*

1. $\sigma(\mathbf{r}) \in C_{1\alpha}(G_j)$, $a_i(\mathbf{r}) \in C_{0\alpha}(G_j)$, $g_i(\mu) \in L_1^{2\alpha}(-1, +1)$ for all $j = 1, 2, \dots, J$, $i = 1, 2, \dots, I$.
2. *There is a ball $K_R(\mathbf{r})$, lying entirely in G_j, such that: a) $a_i \in C_1(K_R(\mathbf{r}))$; b) $\xi_n(\mathbf{r}, \mathbf{s}) \in C_{1\alpha}(K_R \times \Omega)$, $l \geq 0$; c) $D_s^k f \in C_{0\alpha}(K_R)$ on Ω for $k = 0, 1, 2$; d) $\varphi(\mathbf{r}_n, \mathbf{s}) = 0$ for all $\mathbf{r} \in K_R$, $\mathbf{s} \in \Omega$ ($\mathbf{r}_n = \mathbf{r} + \xi_n \mathbf{s}$).*

Then the function $N(\mathbf{r}, \mathbf{s})$, defined by the integral (5), is differentiable with respect to the spatial variables at the point $\mathbf{r}$ for all $\mathbf{s} \in \Omega$.

Let $\mathbf{v}(\mathbf{r}^n)$ be the normal to the surface γ at the point $\mathbf{r}^n = \mathbf{r} + \xi_n \mathbf{s}$, and let $\xi_n < 0$. If, in a neighborhood of the point $\mathbf{r}$, $\gamma \in C_{1\alpha}$ and Δ, δ are sufficiently small positive numbers, then, for $\mathbf{v}(\mathbf{r}^n)\mathbf{s} \neq 0$, $\xi_n \in C_{1\alpha}(K_\Delta(\mathbf{r}) \times K_\delta(\mathbf{s}))$; for $\mathbf{v}(\mathbf{r}^n)\mathbf{s} = 0$ and finite radii of curvature of γ at the point $\mathbf{r}^n$, $\xi_n \in C_{0^{1/2}}(K_\Delta(\mathbf{r}) \times K_\delta(\mathbf{s}))$.

For $\partial N / \partial z$ the formula obtained is

$$\frac{\partial N}{\partial z} = \sum_{i=1}^{I} \left(\frac{\partial a_i}{\partial z} N_i + a_i \frac{\partial N_i}{\partial z} \right),$$

where

$$
\begin{aligned}
N_i &= \int_\Omega g_i(\mathbf{s}\mathbf{s}')\Phi(\mathbf{r},\mathbf{s}')\,d\mathbf{s}', \\
\frac{\partial N_i}{\partial z} &= \int_\Omega d\mathbf{s}'\,(\mathbf{s}'\mathbf{e}_z)\,g_i(\mathbf{s}\mathbf{s}')B(\mathbf{r},\mathbf{s}') \\
&+ \int_{K_R} d\mathbf{r}'\,\frac{\partial'}{\partial z}\left\{\left[B(\mathbf{r}',\mathbf{s}')K(\mathbf{r},\mathbf{r}') - \frac{B(\mathbf{r},\mathbf{s}')}{|\mathbf{r}-\mathbf{r}'|^2}\right]g_i(\mathbf{s}\mathbf{s}')\right\} \\
&+ \int_{G\setminus K_R} d\mathbf{r}'\,\frac{\partial''}{\partial z}\left\{\frac{g_i(\mathbf{s}\mathbf{s}')B(\mathbf{r}',\mathbf{s}')}{|\mathbf{r}-\mathbf{r}'|^2}\exp[-w_\sigma(\mathbf{r},\mathbf{r}')]\right\} \\
&- \int_\gamma d\gamma\,g_i(-\mathbf{s}\mathbf{s}_\gamma)\,\frac{|\mathbf{e}_z\mathbf{v}|}{|\mathbf{r}-\mathbf{r}_\gamma|^2}\,\delta\sigma(\mathbf{r}_\gamma)\int_{|\mathbf{r}_\gamma-\mathbf{r}|}^{d} d\xi\,B(\mathbf{r}+\xi\mathbf{s}_\gamma,-\mathbf{s}_\gamma) \\
&\qquad \times \exp\left[-\int_0^\xi \sigma\,d\xi\right]\left(1-\frac{|\mathbf{r}_\gamma-\mathbf{r}|}{\xi}\right),
\end{aligned}
\tag{6}
$$

$$
\delta\sigma(\mathbf{r}_\gamma) = \sigma_+(\mathbf{r}_\gamma) - \sigma_-(\mathbf{r}_\gamma), \qquad \sigma_\pm(\mathbf{r}_\gamma) = \lim_{\zeta\to 0}\sigma(\mathbf{r}_\gamma \pm \zeta\mathbf{e}_z), \quad \zeta > 0.
$$

The prime $'$ means that differentiation is not performed with respect to the spatial variable $\mathbf{r}$ in the function $B(\mathbf{r},\mathbf{s}')$. The double prime $''$ means that the derivative $\partial\sigma/\partial z$ is computed only at points of the domain G where it is bounded. As $\mathbf{r}$ approaches the surface γ_j, where σ and B have a discontinuity, the last two terms in (6) grow without bound and

$$
\begin{aligned}
\frac{\partial N}{\partial z} &\simeq -\ln|\mathbf{r}-\mathbf{r}_0|\,|\mathbf{e}_z\mathbf{s}_0|\,2\pi g(\mathbf{r},-\mathbf{s}\mathbf{s}_0)\times \\
&\times\left[B_+(\mathbf{r}_0,-\mathbf{s}_0) - B_-(\mathbf{r}_0,-\mathbf{s}_0)\frac{\sigma_+(\mathbf{r}_0)}{\sigma_-(\mathbf{r}_0)}\right],
\end{aligned}
\tag{7}
$$

where

$$
B_\pm(\mathbf{r}_0,-\mathbf{s}_0) = \lim_{\zeta\to 0} B(\mathbf{r}_0 \pm \zeta\mathbf{e}_z,-\mathbf{s}_0), \qquad \mathbf{s}_0 = \frac{\mathbf{r}_0-\mathbf{r}}{|\mathbf{r}_0-\mathbf{r}|}, \qquad \mathbf{s}_\gamma = \frac{\mathbf{r}_\gamma-\mathbf{r}}{|\mathbf{r}_\gamma-\mathbf{r}|},
$$

$\mathbf{r}_0$ is the point of the surface γ nearest to $\mathbf{r}$.

Note that, when the conditions of Theorem 2 are fulfilled, everywhere in G, $|\nabla N| \in L_p(G\times\Omega)$ for any finite p. From (7) one easily obtains an approximate formula for N, valid in neighborhoods of the discontinuity surfaces σ and B. For spherical and plane surfaces, when $g_i = 1/4\pi$, it coincides with the previously obtained formula (6).

The transformation $\Phi(\mathbf{r},\mathbf{s}) = \Phi_0(\mathbf{r},\mathbf{s}) + F(\mathbf{r},\mathbf{s})$ (1) reduces the problem to one with homogeneous boundary conditions and free term $f_0 = f + \hat{g}F$. On the other hand, extracting the first iterations, i.e., putting $\Phi_0 = \Phi_1 + \mathcal{L}^{-1} f_0 = \Phi_2 + \mathcal{L}^{-1} f_0 + \mathcal{L}^{-1}\hat{g}\mathcal{L}^{-1} f_0$, where

$$\mathcal{L}^{-1} f(\mathbf{r},\mathbf{s})\times = \int_{\xi_1}^{0} f(\mathbf{r}+\xi\mathbf{s},\mathbf{s})\times$$

$$\times \exp\left\{\int_{\xi}^{0} \sigma(\mathbf{r}+\xi'\mathbf{s})\, d\xi'\right\} d\xi,$$

we shall have

$$\mathcal{L}\Phi_k = \hat{g}\Phi_k + f_k, \qquad \Phi_k(\mathbf{r}+\xi_{2m+1}\mathbf{s},\mathbf{s}) = \Phi_k(\mathbf{r}+\xi_{2m}\mathbf{s},\mathbf{s}), \quad m = 1, 2, \dots, \mathfrak{N}/2,$$

$$k = 1, 2, \qquad f_1 = \hat{g}\mathcal{L}^{-1} f_0, \qquad f_2 = (\hat{g}\mathcal{L}^{-1})^2 f_0.$$

Thus, the investigation of the solutions of the original problem is reduced to the analysis of F, $\mathcal{L}^{-1} f_0$, and Φ_1; or of F, $\mathcal{L}^{-1} f_0$, $\mathcal{L}^{-1}\hat{g}\mathcal{L}^{-1} f_0$, and Φ_2—problems that are simpler, since F, $\mathcal{L}^{-1} f_0$, $\mathcal{L}^{-1}\hat{g}\mathcal{L}^{-1} f_0$ are quadratures, whose behavior in concrete problems, as a rule, is easily traced, while the study of Φ_1 and Φ_2 is considerably facilitated by the smoothness of the free terms: for $f_0 \in L_\infty(G\times\Omega)$, $\hat{g}\mathcal{L}^{-1} f_0 \in C_{l\alpha}$, while $(\hat{g}\mathcal{L}^{-1})^2 f_0$ satisfies the conditions of Theorem 2.

Theorem 3. Suppose that everywhere in $G \times \Omega$, under the conditions of Theorem 1, the functions Φ and N are defined by the integrals (3) and (5), respectively. Then everywhere in $G \times \Omega$ the derivative $\mathbf{s}\nabla\Phi$ exists; it is uniformly bounded and equals $-\sigma\Phi + B$. If the assumptions of Theorem 2 are fulfilled in $G\times\Omega$, and the point $(\mathbf{r}_*, \mathbf{s}_*)$ is such that, for $\xi_n(\mathbf{r}_*, \mathbf{s}_*) < 0$, the boundary surface γ in a neighborhood of the points $\mathbf{r}_{*n}$ belongs to the class $C_{l\alpha}$ $(l \geq 1, \alpha \geq 0)$, then in the case when the direction $\mathbf{s}_*$ is not tangent, i.e., $\nu(\mathbf{r}_{*n}) \cdot \mathbf{s}_* \neq 0$ for all $\xi_n < 0$, the spatial and angular derivatives of the function Φ at $(\mathbf{r}_*, \mathbf{s}_*)$ exist and are bounded; in a neighborhood of tangent directions, for which $\nu(\mathbf{r}_{*n}) \cdot \mathbf{s}_*$ vanishes for at least some n corresponding to $\xi_n < 0$, the singularities of the derivatives $D_r\Phi$, $D_s\Phi$ have orders respectively $(\mathbf{r} - \mathbf{r}_*)^{-1/2}$, $|\mathbf{s} - \mathbf{s}_*|^{-1/2}$, if the radii of curvature of γ at the points $\mathbf{r}_{*n}$ are finite. On surfaces with zero curvature the angular derivatives of Φ have a singularity $\ln|\mathbf{s} - \mathbf{s}_0|$, where $\mathbf{s}_0$ is the tangent direction, and the function Φ itself has a discontinuity along these directions.

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Note: Figure translations are in progress. See original paper for figures.