The Sample Mean as an Estimate of a Location Parameter under Certain Losses Different from Quadratic

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Abstract

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The Sample Mean as an Estimate of a Location Parameter under Certain Losses Different from Quadratic

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1. We consider the problem of estimating the location parameter $\theta \in R^1$ from the data of a repeated sample $(x_1, \dots, x_n)$ from a population with distribution function (d.f.) $F(x-\theta)$. If $\tilde{\theta} = \tilde{\theta}(x_1, \dots, x_n)$ is proposed as an estimate of θ, then the losses incurred thereby are specified by a loss function (loss) $r(\tilde{\theta}; \theta)$, whose mathematical expectation $R(\tilde{\theta}; \theta) = E_\theta r(\tilde{\theta}; \theta)$ is called the risk of the estimate $\tilde{\theta}$. For a given risk $R(\tilde{\theta}; \theta)$, the notion of admissibility of an estimate is introduced in the natural way.

If the loss is quadratic, $r(\tilde{\theta}; \theta) = (\tilde{\theta} - \theta)^2$, then for $n \geq 3$ the admissibility of the sample mean $\bar{x} = (x_1 + \dots + x_n)/n$ as an estimate of the parameter θ is equivalent to the normality of $F(x)$ (1) (see also (2), which also contains some other results clarifying the role of $\bar{x}$). Here analogous results will be formulated concerning losses different from quadratic.

2. **Estimation under Laplace loss.**
   Theorem. *Let $(x_1, \dots, x_n)$ be a repeated sample of size $n \geq G$ from a population with d.f. $F(x-\theta)$, where $F(x)$ is single-peaked and has a continuously differentiable density $f(x)$ (single-peakedness means the existence of an x_0 such that $f'(x) \geq 0$ for $x \leq x_0$ and $f'(x) \leq 0$ for $x \geq x_0$). Then the admissibility of the sample mean $\bar{x}$ as an estimate of the parameter $\theta \in R^1$ under Laplace loss $r(\tilde{\theta}; \theta) = |\tilde{\theta} - \theta|$ is an exclusive property of the normal law.*

The proof of this theorem is based on combining certain considerations of Pitman (3) with those used in the characterization of distributions by properties of tubular statistics (4,5,7). From the admissibility of $\bar{x}$ under Laplace loss one obtains the relation for $f(x)$

$$\int_{-\infty}^{0} \prod_{1}^{n} f(u + x_i - \bar{x})\, du = \int_{0}^{+\infty} \prod_{1}^{n} f(u + x_i - \bar{x})\, du, \qquad (1)$$

valid for almost all, with respect to the measure $dF(x_1)\cdots dF(x_n)$, values $(x_1,\dots,x_n)$. It is shown that, under the conditions of the theorem, the only probabilistic solution of (1) is the density of the normal law.

The admissibility of the sample mean $\bar{x}$ as an estimate of the population mean from a sample $(x_1,\dots,x_n)$ from a population with normal d.f. $F(x-\theta)$ under Laplace loss follows, for example, from the theorem of Fox and Rubin (6).

3. Let us now consider the loss function

$$r(\tilde{\theta};\theta)=\begin{cases}-\alpha(\tilde{\theta}-\theta), & \tilde{\theta}\le\theta,\\ \beta(\tilde{\theta}-\theta), & \tilde{\theta}\ge\theta,\end{cases} \tag{2}$$

where $\alpha>0,\ \beta>0$ are constants, which generalize the Laplace loss function. Wishing to carry over to the loss (2) the result of the preceding item, one must only bear in mind that for Gaussian quantities $x_1,\dots$

$\cdots$, x_n with $E_\theta x_i=0$, the sample mean $\bar{x}$ is inadmissible as an estimate of θ under loss (2). It is easy to see that $\min E_\theta r(\bar{x}-c;\theta)$ is attained at c', defined by the condition $F(c')=\beta/(\alpha+\beta)$.

Under the conditions of Theorem 2, admissibility of the estimate $\bar{x}-c'$ of the location parameter θ under loss (2) is a characteristic property of the normal law $N(0,\sigma^2)$ for some $\sigma^2\ge 0$.

4. Our considerations are also applicable to loss functions

$$r(\bar{\theta};\theta)=|\bar{\theta}-\theta|^{2m+1}, \tag{3}$$

where m is a natural number. If the sample mean $\bar{x}$ is admissible as an estimate of θ under loss (3), then the density $f(x)$ satisfies, for almost all values $(x_1,\dots,x_n)$ with respect to the measure $dF(x_1)\cdots dF(x_n)$, the relation

$$\int_{-\infty}^{0}|u|^{2m}\prod_{1}^{n}f(u+x_i-\bar{x})\,du=\int_{0}^{+\infty}|u|^{2m}\prod_{1}^{n}f(u+x_i-\bar{x})\,du. \tag{4}$$

It follows easily from (4) that if $f(x)=0$ for $x\ge x_0$ (or for $x\le x_0$), then $f(x)=0$ almost surely with respect to Lebesgue measure. Therefore a continuous unimodal density satisfying (4) is nowhere equal to zero.

In the class of densities $f(x)$ satisfying the following conditions: $f(x)$ is continuously differentiable $2m+1$ times; the functions $\varphi^{(2k)}(x)=\dfrac{d^{2k}}{dx^{2k}}\log f(x)$, $k=1,\dots,m$, preserve their sign; there exists an x_0 such that $\varphi^{(k+1)}(x)\gtreqless 0$ for $x\le x_0$ and $\varphi^{(2k+1)}(x)\lesseqgtr 0$ for $x\ge x_0$, $k=0,1,\dots,m$ (where the choice of the inequality signs may be different for each k), for $n\ge 6$ the only probabilistic solution of equation (4) is the density of the normal law.

The conditions imposed here a priori on $f(x)$ can apparently be substantially weakened. In any case, in the theorem the requirement of unimodality of $f(x)$ can be replaced by unimodality of the function $f(x)e^{ax}$ for some a.

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Note: Figure translations are in progress. See original paper for figures.

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