ASYMPTOTIC PROPERTIES OF SOLUTIONS OF SOME INTEGRAL EQUATIONS

1969

SovietRxiv

Abstract

Full Text

THEORY OF ELASTICITY

V. A. BABESHKO

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF SOME INTEGRAL EQUATIONS

(Presented by Academician V. V. Novozhilov on 10 February 1969)

Mixed problems of the mathematical theory of cracks ($^{1-3}$) lead to an integral equation of the form

$$Kq = \int_{-a}^{a} k(x-\xi)\, q(\xi) d\xi = \pi f(x), \qquad |x| < a, \qquad f(-x) = f(x), \qquad (1)$$

whose kernel has a singularity at zero and tends to a constant at infinity. The asymptotic solution of equation (1) as $a \to 0$ can be constructed by means of N. I. Muskhelishvili' s method of singular integral equations, with subsequent application of the contraction mapping principle to the regularized equation (1). For large values of a, only the zero term of the asymptotics of the solution has been constructed ($^{1-3}$).

In the present note a theorem is given in which an analytic form of the solution of equation (1) as $a \to \infty$ is established, and certain of its properties are also determined. The manner of applying the theorem is illustrated by an example.

We note that integral equation (1) with the indicated properties also arises in mixed problems of elasticity theory for a strip and a wedge with free faces, when a rigid punch acts on them, loaded only by a moment (^{4}).

1. We shall assume that $K(z)$ is an odd function, real on the real axis. Denote by Ω a domain of the form (a parabolic strip) $|\sigma| \leq \infty,\ |\tau| \leq \alpha(\sigma),\ z = \sigma + i\tau$, where $\alpha(\sigma)$ is an even piecewise-smooth function possessing the properties $\alpha(\sigma) \geq \mu > 0,\ \alpha(\sigma) = O(\sigma^{\varepsilon}),\ \sigma \to \infty,\ \varepsilon > 0$. It is assumed that in Ω the function $K(z)$ has no zeros and is regular everywhere, with the exception of the point $z = 0$, where it has a simple pole. On the boundaries Γ_+ (of the upper half-plane) and Γ_- (of the lower half-plane) of the domain Ω the function $K(z)$ is continuous, and in the domain Ω itself the asymptotic estimate holds

$$K(z) = c^2 \left[1 + O(z^{-1})\right], \qquad |z| \to \infty, \qquad \operatorname{Re} z > 0, \qquad k(t) = \int_0^{\infty} K(x) \sin tx \, dx. \qquad (2)$$

In this case (^{5}) the representation

$$zK(z) = K_+(z)K_-(z), \tag{3}$$

is valid, where $K_+(z)$ is regular in the domain $\Omega \cup \operatorname{Im} z > 0$, and $K_-(z)$ in $\Omega \cup \operatorname{Im} z < 0$; moreover,

$$K_+(z) \sim cz^{0.5}, \qquad z \in \Omega, \qquad |z| \to \infty. \tag{4}$$

Introduce for consideration the operator (^{6})

$$F(a,z)\varphi \equiv \frac{1}{2\pi i}\int_{\Gamma_-} \frac{K_-(t)\exp(-2ait)\varphi(t)\,dt}{K_+(t)(t+z)}, \tag{5}$$

acting continuously in the space A of functions $\varphi(z)$, analytic

for $\operatorname{Im} z \le -\delta$ and admitting in $S = \Omega \cap \operatorname{Im} z \le -\delta$ the representation

$$\varphi(z) = \psi(z)z^{-1}, \qquad \max_{z\in S}|\psi(z)| < \infty, \qquad \delta > 0. \tag{6}$$

Denote by E the set of piecewise-smooth contours γ from the domain S, on which, for any function $\varphi(t) \in A$, the identity

$$F(a,z)\varphi \equiv \frac{1}{2\pi i}\int_{\gamma} \frac{K_-(t)\exp(-2ait)\varphi(t)\,dt}{K_+(t)(t+z)}. \tag{7}$$

Set

$$\inf_{t\in\gamma}|\operatorname{Im} t| = \mu_\gamma, \qquad \gamma \in E. \tag{8}$$

Theorem. The solution $q(x)$ of the integral equation (1), satisfying the condition

$$\int_{-a}^{a} q(x)\,dx = 0,$$

is unique in $L_p(-a,a)$ ($p > 1$) and, for an even right-hand side $f(x) \in C_1^\lambda(-a,a)$ ($\lambda > 0.5$), for $a > a_0$ is given by the relation

$$q(x) = \int_{-\infty}^{\infty} \frac{i\Phi(\eta)}{K(\eta)} e^{i\eta x}\,d\eta + \sum_{k=0}^{\infty}(-1)^k[S(a+x) - S(a-x)]F^k(a,z)\psi. \tag{9}$$

Here a_0 is the greatest root of the equation

$$1 = \inf_{\gamma \in E} \max_{z \in \gamma} \frac{e^{-2a_0 \mu_\gamma}}{2\pi} \int_\gamma \left| \frac{z K_-(t) e^{-2a_0 (it - \mu_\gamma)}}{(z+t) K_+(t) t} \right| |dt|. \tag{10}$$

Moreover, the representation

$$q(x) = \omega(x)(a^2 - x^2)^{-0.5}, \qquad \omega(x) \in C(-a, a), \tag{11}$$

is valid, and for $q_n(x)$—the partial sum consisting of n terms of the series (9) and the improper integral—there is the asymptotic estimate

$$[q(x) - q_n(x)](a^2 - x^2)^{0.5} = O\{\exp[-2a(\mu - \varepsilon)n]\} \qquad (a \to \infty),$$

$$x \in [-a, a]; \tag{12}$$

$\varepsilon > 0$ is an arbitrarily small fixed number.

Here the following notation has been introduced:

$$\psi(t) = \int_{-\infty}^{\infty} \frac{\Phi(\eta) e^{-i\eta a}\, d\eta}{K_+(\eta)(t+\eta)}, \qquad S(x) f = \frac{1}{2\pi} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \frac{t f(t) e^{-itx}\, dt}{K_+(t)}, \qquad 0 < \varepsilon < \mu,$$

$$f(x) = \int_{-\infty}^{\infty} \Phi(\eta) e^{i\eta x}\, d\eta, \qquad \Phi(-\eta) = \Phi(\eta). \tag{13}$$

$F^k(a, z)$ is the k-th iteration of the operator $F(a, z)$; $C_1^\lambda(-a, a)$ is the set of functions whose first derivative satisfies a Hölder condition with exponent λ on $[-a, a]$; $C(-a, a)$ is the set of functions continuous on $[-a, a]$.

2. Example. In applications one often encounters the case (1-4), when $f(x) = \cos \eta x$ and $K(z)$ has the form

$$K(z) = z^{-1}(z^2 + D^2)(z^2 + B^2)^{-0.5}, \qquad \operatorname{Im} B = \operatorname{Im} D = 0. \tag{14}$$

We restrict ourselves to the investigation of the case $D > B > 0$.

The function $K(z)$ is regular in the entire complex plane with a deleted neighborhood of zero and with a cut joining, along the imaginary axis, the points-

branches iB and $-iB$ through the point at infinity. The functions $K_+(z)$ and $K_-(z)$, regular in the plane with cuts respectively from $-iB$ to $-i\infty$ and from iB to $i\infty$, are representable in the form

$$K_+(z) = (D - iz)(B - iz)^{0.5}, \quad K_-(z) = (D + iz)(B + iz)^{-0.5}, \tag{15}$$

$$iK_+(z) \sim \sqrt{z}\exp(i\pi/4), \quad K_-(z) \sim i\sqrt{z}\exp(-i\pi/4) \quad (z \to \infty).$$

As the contour Γ_- one may take a contour lying on the left and right banks of the cut connecting the points $-iB$, $-i\infty$. The domain S will be the entire lower half-plane with the indicated cut.

With the indicated choice of the contour Γ_-, the integrand in (5), by virtue of (15), will have a polar singularity at $t = -iD$ on Γ_-, and the integral must be understood in the sense of the principal value.

On the basis of relation (7), as $\gamma \in E$ one may take the family of contours $\gamma(\nu)$ defined by the relation

$$z = \sigma - i(\nu|\sigma| + B), \quad |\sigma| \le \infty, \quad 0 < \nu < \infty. \tag{16}$$

On the contours (16) the integrand in (17) is absolutely summable. Evidently, $2\Phi(x) = \delta(x - \eta) + \delta(x + \eta)$; ($\delta(t)$ is the Dirac delta function). As a result $q(x)$ assumes the form

$$q(x) = -K^{-1}(\eta)\sin\eta x + \varkappa(a + x) - \varkappa(a - x) +$$

$$+ \sum_{k=1}^{\infty}(-1)^k\,[S(a + x) - S(a - x)]F^k(a, z)\psi,$$

$$\begin{aligned}\varkappa(x) = {}&\frac{e^{-Bx}}{\sqrt{\pi x}}\left[\frac{e^{-i\eta a}}{K_+(\eta)} + \frac{e^{i\eta a}}{K_-(\eta)}\right] + \frac{D\sqrt{B - D}\,e^{-Dx}}{D^2 + \eta^2}\ \operatorname{erf}\sqrt{(B - D)x}\,\times\\ &\times\left[\frac{e^{-i\eta a}}{K_+(\eta)(D + i\eta)} + \frac{e^{i\eta a}}{K_-(\eta)(D - i\eta)}\right] - \frac{i\eta e^{i\eta a}\sqrt{B - i\eta}}{K_-(\eta)(D - i\eta)}e^{-i\eta x}\operatorname{erfc}\sqrt{(B - i\eta)x}\\ &+ \frac{i\eta e^{-i\eta a}\sqrt{B + i\eta}}{K_+(\eta)(D + i\eta)}e^{i\eta x}\operatorname{erfc}\sqrt{(B + i\eta)x}.\end{aligned} \tag{17}$$

Let us determine a_0. An upper estimate for a_0 is given by the solution of the inequality

$$1 \ge \min_{0<\nu<\infty}\ \max_{z\in\gamma(\nu)} \frac{1}{2\pi}\int_{\gamma(\nu)}\left|\frac{z(D + it)\sqrt{B - it}\,e^{-2ait}}{(z + t)(D - it)\sqrt{B + it}}\right||dt|. \tag{18}$$

A more overestimated bound is obtained from the inequality

$$\pi \geq (D+B)\,[e^{-2aB} - e^{-2aD}]\,[2aB(D-B)]^{-1} - 2e^{-2aB}\,\mathrm{Ei}[-2a(D-B)]. \quad (19)$$

In the problem of the theory of elasticity on a crack in a strip (1), the case occurs when $B = 0.64;\ D = 1.13$. Solving (19), we find

$$a_0 < 0.3. \quad (20)$$

Thus, relation (17) represents, in the indicated particular case, a solution of equation (1) for the values

$$0.3 \leq a \leq \infty.$$

Rostov State University

Received
25 XII 1968

References Cited

1. V. M. Aleksandrov, B. I. Smetanin, PMM, **29**, issue 4 (1965).
2. V. M. Aleksandrov, B. I. Smetanin, Proceedings of the VI All-Union Conference on the Theory of Plates and Shells, Baku, 1966.
3. B. I. Smetanin, Inzhenern. zhurn., Mekh. tverd. tela, No. 2 (1968).
4. V. M. Aleksandrov, Inzhenern. zhurn., Mekh. tverd. tela, No. 2 (1967).
5. M. G. Krein, UMN, **13**, issue 5 (1958).
6. V. A. Babeshko, DAN, **186**, No. 6 (1969).

Note: Figure translations are in progress. See original paper for figures.