NEW CONDITIONS FOR COMPLETE SOLVABILITY

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Abstract

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NEW CONDITIONS FOR COMPLETE SOLVABILITY

AND QUESTIONS OF REDUCIBILITY

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Let E_x and E_y be finite-dimensional Banach spaces, the first of them real and the second complex. Denote by $L(E_x; E_y)$ the Banach space of all linear operators acting from E_x into E_y, and let $L(E_x; E_y') = L(E_x; L(E_y; E_y))$.

In this paper we study linear multidimensional differential equations of the form

$$Y'(x)h = A(x)hY(x) \qquad (\forall h \in E_x), \tag{1}$$

$$y'(x)h = A(x)hy(x) + f(x)h \qquad (\forall h \in E_x), \tag{2}$$

where $A(x)$ and $f(x)$ are operator functions defined on a convex domain $G_x \subset E_x$ and taking values in the spaces $L(E_x; E_y')$ and $L(E_x; E_y)$, respectively. In the first equation the unknown is an operator function $Y(x)$ with values in $L(E_y; E_y)$, while in the second equation it is a vector function $y(x)$ with values in the space E_y. Recall that a multidimensional differential equation is called *completely solvable* if every Cauchy problem is uniquely solvable.

The paper gives new conditions (necessary and sufficient) for complete solvability of multidimensional differential equations and studies the method of successive approximations; for the case of an almost periodic right-hand side, conditions for complete solvability are formulated in terms of Fourier coefficients and frequencies; with the aid of an estimate for the resolvent of a permutable operator, conditions are indicated for the reducibility of an equation with almost periodic coefficients.

Let $\varphi(x)$ be a continuous operator function defined on the domain G_x and taking values in $L(E_x; E_y)$, and let ∂s_ε be the boundary of the triangle $s_\varepsilon = s(x; \varepsilon h, \varepsilon k)$, whose positive orientation is specified by the order of the vertices $x, x+\varepsilon h, x+\varepsilon k$. Put

$$\operatorname{rot}\varphi(x;h,k)=\lim_{\varepsilon\to+0}\frac{1}{\varepsilon^2}\int_{\partial s_\varepsilon}\varphi(\xi)\,d\xi, \tag{3}$$

if the limit exists. In the case where $\operatorname{rot}\varphi(x;h,k)$ is a bilinear operator in h,k, we write $\operatorname{rot}\varphi(x)hk$. Note that if $\varphi(x)$ is differentiable, then $\operatorname{rot}\varphi(x)hk = \Lambda_{hk}\varphi'(x)hk$ ($\Lambda_{hk}Bhk=(Bhk-Bkh)/2$).

1. Of great interest in the study of multidimensional differential equations is the question of conditions for complete solvability. If the right-hand side is continuously differentiable, then the answer is given by Frobenius' theorem (see, for example, ([1]), p. 356). In the monograph ([2]), R. Nevanlinna gives the condition for complete solvability of the operator equation (1) in the form

$$U(x;h,k)=1+o(\delta^2), \tag{4}$$

where $U(x;h,k)$ is the multiplicative integral of $A(x)$, computed over $\partial s = \partial s(x;h,k)$, and δ is the diameter of the triangle s. Developing this assertion, we arrive at the following theorem.

Theorem 1. Let $A(x)$ and $f(x)$ be continuous. Then complete solvability of equation (2) holds if and only if the conditions ($\forall h,k\in E_x$)

$$\operatorname{rot}A(x)hk=\Lambda_{hk}A(x)hA(x)k, \tag{5}$$

$$\operatorname{rot}f(x)hk=\Lambda_{hk}A(x)hf(x)k. \tag{6}$$

The proof is based on the formulas

$$U(x;h,k)=1+\int A(\xi)\,d\xi-\Lambda_{hk}A(x)hA(x)k+o(\delta^2), \tag{7}$$

$$\int_{\partial s} f(\xi)\,d\xi=\Lambda_{hk}A(x)hf(x)k+o(\delta^2), \tag{8}$$

which hold for any compact set $K\subset G_x$, the above-cited assertion of R. Nevanlinna and one result of V. Shapiro ([3]).

We shall seek a solution of equation (2) satisfying the initial condition $y(\xi)=\eta$, by the method of successive approximations

$$y_0(x)\equiv\eta$$

$$y_p'(x)h=A(x)hy_{p-1}(x)+f(x)h,\qquad y_p(\xi)=\eta \tag{9}$$

$$(p = 1, 2, \ldots).$$

Theorem 2. The method of successive approximations (9) is realizable for arbitrary initial conditions if and only if $(\forall h, k \in E_x)$

$$\operatorname{rot} A(x) = 0, \qquad \Lambda_{hk} A(x) h A(x) k = 0; \tag{10}$$

$$\operatorname{rot} f(x) = 0, \qquad \Lambda_{hk} A(x) h f(x) k = 0. \tag{11}$$

Let $U(A; f)$ denote the totality of all continuous vector functions $u(x)$ for which the curvilinear integral of $\varphi(x)h = A(x)hu(x) + f(x)h$ is path-independent. It can be shown that $U(A; f)$ contains all constants and is mapped into itself by the integral operator

$$I_\eta u(x) = \eta + \int_\xi^x A(\zeta)\, d\zeta\, u(\zeta) + \int_\xi^x f(\zeta)\, d\zeta \tag{12}$$

for any $\eta \in E_y$, if and only if conditions (10) and (11) are satisfied.

2. Let us now consider equation (2) with almost periodic operator functions $A(x)$ and $f(x)$, and let

$$A(x) \sim \sum A_\lambda e^{i\lambda x}, \qquad f(x) \sim \sum f_\lambda e^{i\lambda x}; \tag{13}$$

in this case $G_x = E_x,\ \lambda \in E_x^*$ (see (7)).

Theorem 3. Complete solvability for equation (2) with almost periodic operator functions $A(x)$ and $f(x)$ holds if and only if $(\forall h, k \in E_x)$

$$\Lambda_{hk} i\lambda h\, A_\lambda k = \Lambda_{hk} \left(\sum_{\mu+\nu=\lambda} A_\mu h A_\nu k \right), \tag{14}$$

$$\Lambda_{hk} i\lambda h\, f_\lambda k = \Lambda_{hk} \left(\sum_{\mu+\nu=\lambda} A_\mu h f_\nu k \right). \tag{15}$$

We emphasize that no smoothness of $A(x)$ and $f(x)$ is assumed in the theorem. The proof of Theorem 3 is based on the use of Theorem 1 and the following assertion (the theorem on the rotor).

Theorem 4. Let $f(x)$ be an almost periodic function, and suppose that one can specify an almost periodic function $R(x)$ with values

in $L_2(E_x; E_y)$, such that for any $h, k \in E_x$

$$R(x)hR \sim \sum \Lambda_{hk}(i\lambda hf, \lambda k)e^{i\lambda x}. \tag{16}$$

Then

$$\int_{\partial s} f(\xi)\, d\xi = \iint_s R(\xi)\, d\xi_1\, d\xi_2. \tag{17}$$

In formula (17), on the right there is the affine integral (see ([2])); $L_2(E_x; E_y)$ is the space of bilinear operators defined on $E_x \oplus E_x$ and with values in E_y. Theorem 4 is most simply proved with the aid of Bochner–Fejér polynomials (see, for example, ([4])), whose theory is not difficult to carry over to vector-valued (and operator-valued) functions.

3. An operator $A \in L[E_x; E_y]$ is called **permutable** if $\Lambda_{hk} AhAk = 0$. A linear functional λ is called an **eigenfunctional** of the operator A if $Ah\xi = (\lambda h)\xi$ $(0 \neq \xi \in E_y)$ (see ([5])). The totality $\sigma(A)$ of all eigenfunctionals is called the **spectrum** of the operator A. The least natural number p for which $(\lambda h - Ah)^{p+1} y = 0$ implies $(\lambda h - Ah)^p y = 0$ is called the **index** $n(\lambda)$ of the functional λ.

It can be shown that $\lambda \in \sigma(A)$ if and only if $\det(\lambda h - Ah) = 0$. In expanded form this equation has the form

$$(\lambda h)^n - p_1(h)(\lambda h)^{n-1} + \dots + (-1)^n p_n(h) = 0, \tag{18}$$

where $p_j(h)$ is a homogeneous functional of degree j. In order to eliminate h from the written equation, call the **product** $f\varphi$ of symmetric functionals f and φ the symmetric functional uniquely determined by the formula $(f\varphi)h^{p+q} = (fh^p)(\varphi h^q)$ (f and φ are assumed to be p- and, respectively, q-linear functionals). Let p_j be the symmetric j-linear functional uniquely determined from the relation $p^j_{jh} = p_j(h)$ (as above, we use here the result from ([6]), p. 769). Then equation (18) takes the form

$$\mathfrak{D}(\lambda) \stackrel{\text{def}}{=} \lambda^n - p_1\lambda^{n-1} + \dots + (-1)^n p_n = 0. \tag{19}$$

Let us note that for a permutable operator the formula

$$\mathfrak{D}(\lambda) = \prod_{\mu \in \sigma(A)} (\lambda - \mu)^{n(\mu)} \tag{20}$$

is valid.

If $\lambda \notin \sigma(A)$, then the equation $(\lambda h - Ah)y = fh$ is uniquely solvable if and only if $\Lambda_{hk}(\lambda h - Ah)fk = 0$ $(f \in L(E_x; E_y))$. The inverse operator $R(\lambda; A)$ will be called the **resolvent** of the permutable operator A.

Theorem 5. Let A be a permutable operator and $\lambda \notin \sigma(A)$. Then

$$\|R(\lambda; A)\| \leq c(A) \left(\sum_{j=1}^{n(A)} \frac{1}{[\rho(\lambda)]^j} \right), \tag{21}$$

where $\rho(\lambda)$ is the distance from λ to $\sigma(A)$; $n(A)$ is the maximum of the indices of the eigenfunctionals of the operator A.

Let us now consider the operator $\mathfrak{C} \in L\{E_x; E_y\}$, given by the formula

$$\mathfrak{C} hY = AhY + YBh, \tag{22}$$

where $L\{E_x; E_y\} = L[E_x; L(E_y; E_y)]$ and $A, B \in L[E_x; E_y]$, $Y \in L(E_y; E_y)$.

Theorem 6. The operator $\mathfrak{C}$ is permutable if and only if the operators A and B are permutable; $\sigma(\mathfrak{C}) = \sigma(A) + \sigma(B)$, and

$$n(\nu) \leq \max_{\lambda+\mu=\nu} [n(\lambda) + n(\mu)] - 1,$$

where $\lambda \in \sigma(A)$, $\mu \in \sigma(B)$, and $\nu \in \sigma(\mathfrak{C})$.

4. Consider the operator equation, completely solvable for all ε,

$$Y'(x)h = (A + \varepsilon B(x))hY(x), \tag{23}$$

where $A \in L[E_x; E_y]$, and the operator function $B(x)$ is an almost periodic function, $B(x) \sim \sum B_\mu e^{i\mu x}$, with $B_0 = 0$. The operator A is permutable, and suppose that its spectrum consists of the functionals $\lambda^1, \dots, \lambda^p$. Denote by σ_B the spectrum of the almost periodic function $B(x)$, and let $j\sigma_B$ be the set of all sums of the form $\mu^1 + \dots + \mu^j$, where $\mu^i \in \sigma_B$ $(i = 1, \dots, j)$. Suppose that $\|B()\|_* = \sum \|B_\mu\| < \infty$ and

$$\delta_j = \inf \|i\nu - (\lambda^t - \lambda^s)\|, \tag{24}$$

where $\nu \in j\sigma_B$ and $t, s = 1, 2, \dots, p$. Put

$$\Delta_j = \sum_{k=1}^{2n(A)-1} \frac{1}{\delta_j^k}, \tag{25}$$

$$\frac{1}{\varkappa} = c(\mathfrak{C}) \|B()\|_* \lim_{p\to\infty} (\Delta_1 \cdots \Delta_p)^{1/p}, \tag{26}$$

where $c(\mathfrak{C})$ is the constant in an estimate of type (21) for the resolvent of the operator $\mathfrak{C}$, where $\mathfrak{C}hY = AhY + YBh$.

Theorem 7. *Under the assumptions stated above, the solution of equation (23) with $Y(0,\varepsilon) = 1$ is representable in the form*

$$Y(x,\varepsilon) = Z(x,\varepsilon)e^{Ax}, \tag{27}$$

where the series

$$Z(x,\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j Z_j(x) \tag{28}$$

converges uniformly and absolutely for $|\varepsilon| < \varkappa$, and each $Z_j(x)$ is an almost periodic operator function whose spectrum lies in $j\sigma_B$.

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Note: Figure translations are in progress. See original paper for figures.