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Abstract

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ON A LIMIT THEOREM FOR THE NUMBER OF CLASSES OF PURELY RADICAL QUADRATIC FORMS OF NEGATIVE DETERMINANT

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The limit theorem for the function $h(-D)$ (the number of classes of purely radical quadratic forms of negative determinant $-D$) was derived by M. B. Barban ([1,2]) from asymptotic formulas obtained by him for the moments of this function.

A generalization of Esséen' s inequality, recently found by the author ([3]), makes it possible in the present note to prove the following refinement of the indicated limit theorem.

Theorem. For $x > 0,\ N \to \infty$,

$$\frac{1}{N} \sum_{\substack{D \le N \\ h(-D) \le \frac{2}{\pi}\sqrt{Dx}}} 1 = \Phi(x) + O\left(\frac{\ln\ln N\,(\ln\ln\ln N)^{1/2}}{(\ln N)^{1/2}}\right) \tag{1}$$

uniformly in x, where $\Phi(x) = F(\ln x)$; $F(u)$ is the distribution function defined by the characteristic function

$$f(\xi) = \prod_{p>2}\left(\frac{1}{p} + \frac{1}{2}\left(1-\frac{1}{p}\right)\left\{\left(1-\frac{1}{p}\right)^{-i\xi} + \left(1+\frac{1}{p}\right)^{-i\xi}\right\}\right). \tag{2}$$

We shall precede the proof by several lemmas.

Lemma 1. Let $\chi_D(n) = (-D/n)$ be the Jacobi symbol, if $(n, 2D) = 1$; $\chi_D(n) = 0$ otherwise. As $N \to \infty$,

$$\frac{1}{N}\sum_{D \le N} L^{i\xi}(1, \chi_D) = f(\xi) + O\left(\exp\left\{-a\frac{\ln N}{\ln\ln N} + b|\xi|\ln\ln\ln N\right\}\right) \tag{3}$$

uniformly in ξ for all real ξ. Here $f(\xi)$ is defined by formula (2), and a and b are positive constants.

The proof follows the scheme of work ([1]), with a small modification connected with nonintegral exponents and the uniformity of the estimate in ξ. In doing so one uses the density theorem of E. Bombieri ([4]), a consequence of which, as M. B. Barban showed ([5]), is the absence of zeros of Dirichlet L-series corresponding to primitive characters, for "almost all" moduli in the region $\sigma \geq 1-\alpha$, $|t| \leq k^{\beta}$, where α and β are positive constants.

An immediate consequence of Lemma 1 is the limit theorem for $L(1, \chi_D)$.

Lemma 2. Let $F(x)$ and $G(x)$ be distribution functions, and let $f(\xi)$ and $g(\xi)$ be their characteristic functions. For $T > 0$,

$$\sup_x |F(x) - G(x)| < C_1 \left\{ Q_F\left(\frac{1}{T}\right) + \int_0^T \left| \frac{f(\xi) - g(\xi)}{\xi} \right| d\xi \right\}, \tag{4}$$

where C_1 is an absolute constant, $Q_F(l) = \sup_x \{F(x+l) - F(x)\}$.

The proof is given in paper ([3]).

Lemma 3. Let $\eta_1, \eta_2, \dots, \eta_n$ be independent random variables,

$$\eta = \eta_1 + \eta_2 + \dots + \eta_n, \qquad Q_m(l) = \sup_x P(x < \eta_m \leq x + l).$$

Then

$$Q(l) = \sup_x P(x < \eta \leq x + l) < C_2 \left[\sum_{m=1}^{n} (1 - Q_m(l)) \right]^{-1/2}, \tag{5}$$

where C_2 is an absolute constant.

This inequality for concentration functions is due to B. A. Rogozin ([6]).

We can now prove the theorem formulated at the beginning. From formula (2) it follows that $F(x)$ is the distribution function of the sum of independent random variables η_p (p runs through the odd primes), taking the three values 0, $\ln(1-1/p)^{-1}$, and $\ln(1+1/p)^{-1}$ with probabilities respectively $1/p$, $\frac{1}{2}(1-1/p)$, and $\frac{1}{2}(1-1/p)$. Hence, by Lemma 3,

$$Q_F(l) = O\left(\sqrt{l \ln \frac{1}{l}}\right). \tag{6}$$

We apply Lemma 2, using, for the estimate of the difference of characteristic functions in a neighborhood of zero, the known estimate $\ln L(1, \chi_D) = O(\ln D)$, and outside this neighborhood—Lemma 1. We obtain, for $\delta > 0$:

$$\sup_x \left| F(x) - \frac{1}{N} \sum_{\substack{D \leq N \\ \ln L(1,\chi_D) < x}} 1 \right| \ll \sqrt{\frac{\ln T}{T}} + \delta \ln N +$$

$$+\ln\frac{T}{\delta}\exp\left\{-a\frac{\ln N}{\ln\ln N}+bT\ln\ln\ln N\right\}.$$

The proof is completed by choosing

$$\delta = 1/\ln^2 N, \qquad T = a\ln N/2b\ln\ln N\cdot\ln\ln\ln N,$$

if one takes into account Gauss' s well-known formula

$$h(-D) = \frac{2}{\pi}\sqrt{D}\,L(1,\chi_D)$$

for $D > 1$.

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Note: Figure translations are in progress. See original paper for figures.