ON PERFECTLY NORMAL BICOMPACTS

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Abstract

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ON PERFECTLY NORMAL BICOMPACTS

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This note gives answers to a number of questions connected with perfectly normal bicompacts. In I and III examples are constructed that are related to open mappings of nonmetrizable perfectly normal bicompacts onto compacta. In II a locally linearly connected, linearly connected nonmetrizable perfectly normal bicompact is constructed. In IV it is proved that there is no universal perfectly normal bicompact. A minor inconvenience is the use in II and III of the continuum hypothesis.

I. In ([3]) N. V. Velichko constructed a nonmetrizable bicompact which is mapped openly and compactly onto an interval. Recently M. Choban constructed a countably multiple mapping of a nonmetrizable bicompact onto a compactum. We shall construct an open countably multiple mapping of a nonmetrizable perfectly normal bicompact onto the Cantor perfect set.

Let M_1 be the set of points of the real line consisting of zero and points of the form $1/n$, where $n = \pm 1, \pm 2, \ldots$. Let K be the Cantor perfect set lying on an interval, consisting of numbers in whose ternary notation there are no ones. This set is constructed by successively removing from the interval finite collections of intervals ([1]), and the whole construction is carried out in a countable number of steps. To each point k of the Cantor perfect set which is not an endpoint of an interval removed at some step, we put in correspondence sequences $a_n^+(k)$ and $a_n^-(k)$ of adjacent intervals as follows: $a_n^+(k)$ is the interval lying to the right of k, nearest to k, among the intervals removed at the n-th step or earlier; $a_n^-(k)$ is the interval lying to the left of k, nearest to k, among the intervals removed at the n-th step or earlier. For many k these sequences are defined by us not for all n, but only starting from some one. In order to avoid ambiguities, we shall take as the missing terms of the sequence the improper intervals of the real line $(\max K, +\infty)$ or, respectively, $(-\infty, \min K)$.

To a point k of the Cantor perfect set we put in correspondence two sets $P^+(k)$ and $P^-(k)$ of the product $K \times M_1$. A point $x \in K \times \{1/n\} \subset K \times M_1$, $n = 1, 2, \ldots$, is assigned to $P^+(k)$ if x lies to the right of the interval $a_n^+(k)$, and to $P^-(k)$ in the opposite case. If $x \in K \times \{1/n\}$, $n = -1, -2, \ldots$, then we assign the point x to $P^+(k)$ if x lies to the right of the interval $a_n^-(k)$, and to $P^-(k)$ in the opposite case. If $x \in K \times \{0\}$, then we assign the point x to $P^+(k)$ if x lies to the right of k, and to $P^-(k)$ in the opposite case. The sets $P^+(k)$ and $P^-(k)$ are open in $K \times M_1$, do not intersect, and cover all points of $K \times M_1$ with the exception of $(k, 0)$.

In the set $K \times M_1$ we replace the points of $K \times \{0\}$ which are not endpoints of adjacent intervals by pairs of points. By k^+ and k^- we shall denote the two points which we have obtained from $(k, 0)$. The inverse transformation of the resulting set X into $K \times M_1$, i.e. the identification of the points k^+ and k^- into one point k (and elsewhere the identity), will be denoted by φ.

We define a topology on X. If in the construction of X the point x was not subjected to any splitting, then as its neighborhood we take the set $\varphi^{-1}(U)$, where U is a neighborhood of the point $\varphi(x)$ in $K \times M_1$. Then as neighborhoods of the points k^+ and k^- we shall take, respectively, the sets $k^+ \cup \varphi^{-1}(U \cap P^+(k))$ and

$k - \bigcup \varphi^{-1}(U \cap P^-(k))$, where U is a neighborhood of the point $k = \varphi(k^+) = \varphi(k^-)$ in $K \times M_1$. It is not hard to verify that the topology has been defined correctly and that in it the space X is a bicompactum. The subspace $\varphi^{-1}(K \times \{0\})$ is homeomorphic to the "two arrows" ([2]), and therefore is perfectly normal and nonmetrizable. The complement of this subspace consists of a countable number of copies of the Cantor perfect set, and consequently the whole space X is represented as the union of a countable number of its closed perfectly normal subsets and, hence, is itself perfectly normal, while, containing a nonmetrizable subspace, it is nonmetrizable.

Let $\pi : K \times M_1 \to K$ be the projection onto the factor, $f = \pi\varphi : X \to K$. As is easy to see, the mapping f is open and the preimage of each point consists of a countable number of points.

II. Now, assuming that $2^{\aleph_0} = \aleph_1$, we shall construct a locally connected nonmetrizable perfectly normal bicompactum*.

Lemma. *Under the assumption that $2^{\aleph_0} = \aleph_1$, in every uncountable compactum there is an uncountable set whose intersection with any nowhere dense set is at most countable.*

The proof of the lemma may be found in ([4]), p. 534. The required set is a Luzin v-space.

Let Kv be the ordinary square, and let M be the set lying in it whose existence is established by the lemma. For simplicity we shall assume that M does not meet the boundary of the square. We shall replace the points of the set M by circles (by this we mean the following: take a family of (pairwise disjoint) sets $\{Y^\alpha\}_{\alpha \in Kv}$, indexed by the points of the square, where Y^α is a circle if $\alpha \in M$, and a point otherwise. Then $Y = \bigcup_{\alpha \in Kv} Y^\alpha$ is the required set). Let $\varphi : Y \to Kv$ be the mapping of the resulting set Y onto Kv, again gluing these circles into points (i.e., each set Y_α is sent to the point indexing it). The circle Y_α may be understood as the set of rays emanating from the point $\alpha \in M$. This representation will help us define a topology on the set Y. If a point $y \in Y$ is such that $\varphi(y)$ does not belong to M, then as its neighborhoods we take sets $\varphi^{-1}(U)$, where U is a neighborhood of the point $\varphi(y)$. Consider the case when $\varphi(y) = \alpha \in M$. Let V be a neighborhood of the point y on Y_α; let V^* be the union of all rays issuing from α (without the point α itself) corresponding to the points of the set V. A neighborhood of the point y will be the set $\varphi^{-1}(V^* \cap U) \cup V$, where U is a neighborhood of the point $\varphi(y)$. As is easy to see, we have correctly defined a certain topology, and in this topology the set Y is a bicompactum (in order to be convinced of the latter, it suffices to check that the mapping is perfect).

We shall show that the bicompactum Y is not metrizable. For this, first note that if a sequence $\{y_n\}$ converges to a point y, then the sequence of bicompacts $\{\varphi^{-1}\varphi(y_n)\}$ also converges to the same point. It follows from this that, for any metric, the set of those compacta $\{\varphi^{-1}(x)\}_{x\in M}$ whose diameter is greater than $\varepsilon > 0$ is finite, which is impossible in view of the uncountability of M.

We shall show that the space Y is hereditarily finally compact. For this, obviously, it suffices to show that into any family Γ of open sets in the space Y one can insert a countable family with the same union. First of all, for those sets of the form $\varphi^{-1}(x)$, $x \in Kv$, for which this is possible, choose neighborhoods of the form $\varphi^{-1}(U)$, inserted in elements of the family Γ. Let W be the open subset of the square which is the union of all such sets U; $\{U_1, U_2, ...\}$ is a countable covering of the set by open sets whose preimages lie

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* The problem of constructing such a bicompactum was posed by V. I. Ponomarev. We note that, by virtue of one theorem of Mardešić, this bicompactum cannot be zero-dimensionally mapped onto a compactum.

in elements of Γ, which exists in view of the final compactness of subsets of the square,

$$\Gamma^* = \{\varphi^{-1}(U_1),\ \varphi^{-1}(U_2), ...\}.$$

The boundary of the set W is nowhere dense, and therefore cuts out from M no more than a countable part M_0. The set $\varphi^{-1}(M_0)$, as the union of a countable number of compacta, is hereditarily finally compact, and therefore from Γ we can choose a countable number of elements Γ_0, whose union cuts out from $\varphi^{-1}(M_0)$ the same set as does the union of all elements of Γ. If a point y is covered by an element of the family Γ, then, obviously, $\varphi(y) \in [W]$. If $\varphi(y) \in W$, then y is covered by an element of the family Γ^*. If $\varphi(y)$ belongs to the boundary of W, then $\varphi(y) \in M$ and y is covered by an element of the family Γ_0.

Thus the family $\Gamma^* \cup \Gamma_0$ has the same union as Γ. It remains only to note that the family $\Gamma^* \cup \Gamma_0$ is no more than countable.

As is easy to see, the bicompactum Y is connected and locally connected.

III. N. V. Velichko ([3]) constructed a nonmetrizable bicompactum mapping openly onto an interval in such a way that the preimage of each point is also an interval. We shall now construct the same kind of example, adding to the listed properties perfect normality of the bicompactum.

In the bicompactum Y constructed in the preceding item, perform the following identifications. In the circles Y'_α, $\alpha \in M$, identify points symmetric with respect to the vertical diameter. The resulting bicompactum Y_1 is nonmetrizable by the same considerations as Y. It is perfectly normal, as a continuous image of a perfectly normal bicompactum. The mapping φ naturally generates a mapping

$$\varphi_1 : Y_1 \to \mathrm{Kv}.$$

Let $\pi_1 : \mathrm{Kv} \to I$ be the projection of the square onto the horizontal interval,

$$f_1 = \pi_1 \varphi_1 : Y_1 \to I.$$

As is easy to see, the mapping f is open and the preimage of each point is homeomorphic to an interval.

IV. As is known ([1]), every (metric) compactum is homeomorphic to some closed subspace of the Hilbert cube, and the Hilbert cube itself is also a compactum. The question naturally arises whether there exists a perfectly normal bicompactum Y such that every perfectly normal bicompactum is homeomorphic to some (closed) subset of the bicompactum Y^*. We shall now obtain a negative answer to this question as a consequence of the following assertion.

Assertion. *There exists a family C of cardinality $2^{2^{\aleph_0}}$ of pairwise nonhomeomorphic perfectly normal bicompacta.*

Proof. Let S be the naturally linearly ordered "two arrows" ([2]), let $\lambda : S \to I$ be the mapping of the "two arrows" onto the interval $[0, 1]$, gluing pairwise the ends of the empty intervals,

$$r^- = \min \lambda^{-1}(r), \qquad r^+ = \max \lambda^{-1}(r).$$

Let M be some subset of the set of irrational points of the interval. In the product $S \times I$, glue pairwise the points $(r^-, 1)$ and $(r^+, 1)$ for all $r \in I$, and the points $(r^-, 0)$ and $(r^+, 0)$ for $r \in M$. The bicompactum $Z(M)$ obtained as a result of this factorization ψ_M, obviously, is perfectly normal. The mapping $\lambda\pi : S \times I \to I$, where

$$\pi : S \times I \to S$$

is the projection onto the factor, generates a mapping

$$F_M : Z(M) \to I.$$

The preimage of a point under this mapping will be called a layer. The layer $F_M^{-1}(r)$ is homeomorphic to a circle if $r \in M$, and to an interval otherwise.

Two subsets of the irrational numbers of the interval we shall call similar if there exists a homeomorphism of the interval onto itself carrying one set into the other.

As is easy to see, the similarity thus introduced is a genuine equivalence relation, and we obtain—

* The problem in this form was posed by V. I. Ponomarev. The idea of solving this problem by constructing a family of cardinality greater than $2^{\aleph_0}$ of pairwise nonhomeomorphic perfectly normal bicompacta was suggested by A. V. Arhangel' skii.

we obtain a partition of the set of all subsets of the interval into equivalence classes. As is easy to see, in each class there are no more than $2^{\aleph_0}$ members (for such is the cardinality of the set of homeomorphisms). If one takes into account that the total number of subsets of the set of irrational numbers is $2^{2^{\aleph_0}}$, then it follows that there are the same number of equivalence classes.

We shall show that the spaces $Z(M_1)$ and $Z(M_2)$ are homeomorphic if and only if the sets M_1 and M_2 are similar. When the sets are similar, a homeomorphism of the bicompacts is constructed in an obvious way, so that in one direction the assertion is trivial. Let us turn to the converse implication.

Let l be a subset of $Z(M)$ homeomorphic to an interval. Then, evidently, either l lies in one of the layers, or l splits into the union of three sets l_1, l_2, l_3, homeomorphic to an interval (or, possibly, to a point), of which the end ones lie in layers, while l_2 lies in $\Psi_M(S \times \{1\})$.

Let $G : Z(M_1) \to Z(M_2)$ be a homeomorphism. From the observation made above and from the fact that the set of points that separate $Z(M_i)$ is everywhere dense in any interval lying in $\Psi_{M_i}(S \times \{1\})$, which cannot be said of intervals lying in layers, it follows easily that each layer is mapped homeomorphically onto some layer; moreover, since a circle is not homeomorphic to an interval, the layers over points of M_1 pass to layers over points of M_2. We note also that (from the same considerations) G maps $\Psi_{M_1}(S \times \{1\})$ homeomorphically onto $\Psi_{M_2}(S \times \{1\})$. From what has been said it follows easily that the composite mapping

$$I \underset{F_{M_1}}{\simeq} \Psi_{M_1}(S \times \{1\}) \underset{G}{\simeq} \Psi_{M_2}(S \times \{1\}) \underset{F_{M_2}}{\simeq} I$$

establishes a similarity between M_1 and M_2.

Now, choosing one representative from each equivalence class and constructing the corresponding spaces $Z(M)$, we obtain the required family C.

Corollary. *There is no perfectly normal bicompact Y such that every perfectly normal bicompact is homeomorphic to some subspace of the bicompact Y.*

Proof. Suppose that this is not so. Then, since a perfectly normal bicompact has no more than $2^{\aleph_0}$ closed subsets ([2]), in any family of perfectly normal bicompacts of larger cardinality there will be homeomorphic ones, which contradicts the assertion.

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Note: Figure translations are in progress. See original paper for figures.