ON THE NUMBER OF ENTRIES OF TRAJECTORIES OF A MARKOV PROCESS INTO A GIVEN SET OF THE PHASE SPACE

MATHEMATICS

1969

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Abstract

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UDC 519.217

MATHEMATICS

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ON THE NUMBER OF ENTRIES OF TRAJECTORIES OF A MARKOV PROCESS INTO A GIVEN SET OF THE PHASE SPACE

(Presented by Academician A. N. Kolmogorov, 20 XII 1968)

Let ξ_t, $-\infty < t < +\infty$, be a stationary, time-homogeneous Markov process with values in the measurable space $[\mathcal{E}, \mathfrak{B}_{\mathcal{E}}]$.

Definition 1. We shall say that the time t is an **entry of a trajectory** of the process ξ_t into B from $\bar{A}$, if $\xi_t \in B \subseteq A$ and, for any $\varepsilon > 0$, there exist $s \in (t-\varepsilon, t)$ such that $\xi_s \in \bar{A} = \mathcal{E} \setminus A$.

By $\eta_A(\Delta, B)$ we denote the number of entries of ξ_t into B from $\bar{A}$ that occurred in the time interval Δ, assuming that $\eta_A(\Delta, B)$ are random variables. The random variables $\eta_A(\Delta, B)$ form a stationary random stream (^{1}), whose characteristics (intensity, moments, etc.) can be expressed in terms of the function $M_{x,A}(\Delta, B)$, defined for $\Delta \subset R^+ = (0, +\infty)$ by the relation

$$M_{x,A}(\Delta, B) = M\{\eta_A(\Delta, B) \mid \xi_0 = x\}. \tag{1}$$

Below we shall call this function **basic**. Suppose that $M_{x,A}(\Delta, A) < \infty$ for every interval $\Delta \subset R^+$ having finite length $|\Delta| < \infty$. This assumption is essential; however, it is satisfied for a wide class of Markov processes with a discrete intervention of chance (^{2}) and sets $B \subseteq A \subset \mathcal{E}$. Under the assumption made, $M_{x,A}(\Delta, B)$ can be extended to a measure on $[A \times R^+, \mathfrak{B}_A \times \mathfrak{B}_{R^+}]$, where $\mathfrak{B}_A = \mathfrak{B}(B : B \subseteq A,\ B \in \mathfrak{B}_{\mathcal{E}})$, $\mathfrak{B}_{R^+}$ is the σ-algebra of Borel sets on R^+. The basic functions satisfy the relation

$$M_{x,A}(\Delta + t, B) = \int_{y \in \mathcal{E}} P_x(t, dy)\, M_{y,A}(\Delta, B), \tag{2}$$

where $\Delta + t = \{s : \ s - t \in \Delta\}$, $\Delta \subset R^+$, and $P_x(t, A)$ are the transition probability functions.

Definition 2. A family of measures P_t on $[\mathcal{E}, \mathfrak{B}_{\mathcal{E}}]$ **converges**, as $t \to \infty$, **to** P **with respect to the partition** $\mathcal{C} = \{\mathcal{E}_{n,k}\}$, $\mathcal{E} = \bigcup_k \mathcal{E}_{n,k}$, $\mathcal{E}_{n,k} \cap \mathcal{E}_{n,l} = \emptyset$, $k \neq l$, $\mathcal{E}_{n,k} = \bigcup_{l \in I_{n,k}} \mathcal{E}_{n+1,l}$, if for every n

$$\lim_{t\to\infty} \sum_k \left| P_t(\mathcal{E}_{n,k}) - P(\mathcal{E}_{n,k}) \right| = 0. \tag{3}$$

From the convergence of P_t to P in variation (^{3}) there follows convergence with respect to any partition $\mathcal{C}$.

Theorem 1. Let $P_x(t, A)$ converge to the stationary measure $P_\infty(A) = P\{\xi_t \in A\}$ with respect to $\mathcal{C}$, and let the basic functions be such that

$$\lim_{n\to\infty} \max_k \left[\sup_{x \in \mathcal{E}_{n,k}} M_{x,A}(\Delta_0 + t_0, B) - \inf_{x \in \mathcal{E}_{n,k}} M_{x,A}(\Delta_0 + t_0, B) \right] = 0, \tag{4}$$

$$M_{x,A}(\Delta_0 + t_0, B) \leq g(x),$$

$$\lim_{t\to\infty} \int_{y\in\mathcal{E}} P_t(x, dy)\, g(y) = \int_{\mathcal{E}} P_\infty(dy)\, g(y).$$

Then

$$\lim_{t\to\infty} M_{x,A}(\Delta_0 + t, B) = \mu_A(B) = M\eta_A(\Delta_0, B), \tag{5}$$

where $\Delta_0 = (0, 1)$.

Proof follows from the uniform integrability of $M_{y,A}(\Delta_0 + t_0, B)$, the approximation of the main function by step functions on $\mathcal{C}$, and relations (2), (3).

Relation (5) is an analogue of Blackwell' s theorem (4). The main difficulty in using Theorem 1 consists in verifying the convergence of $P_t(x, A)$ to $P_\infty(A)$ with respect to partitions $\mathcal{C} = \{\mathcal{C}_{n,k}\}$ that are "sufficiently" fine for (4).

If $\Delta_i = (t_i, t_i + u_i)$, $t_i + u_i < t_{i+1}$ $(\Delta_i < \Delta_{i+1})$, $i = 1, \ldots, k$, then to the k-dimensional rectangle $\Delta_1 \times \cdots \times \Delta_k$ we assign the number

$$M_{(k),A}(\Delta_1 \times \cdots \times \Delta_k, B) =$$

$$= \int_{y_1 \in B} \int_{s_1 \in \Delta_1} \mu_A(dy_1)\, ds_1 \int_{y_2 \in B} \int_{s_2 \in \Delta_2 - s_1} M_{y_1,A}(ds_2, dy_2) \cdots$$

$$\cdots \int_{y_k \in B} \int_{s_k \in \Delta_k - (s_1 + \cdots + s_{k-1})} M_{y_{k-1},A}(ds_k, dy_k).$$

We shall assume that $M_{(k),A}$ is bounded for any bounded intervals $\Delta_i < \Delta_{i+1}$, $i = 1, \dots, k$. In this case it can be extended to a measure on the Borel sets of the cone $C_k = \{(t_1, \dots, t_k),\ t_1 < t_2 < \dots < t_k\}$. Using permutations of the values of the coordinates of the points of C_k, one can extend $M_{(k),A}$ to a measure in R^k. The introduced measure has the property that, for any Borel set $D \subset R^k$, the measure $M_{(k),A}(D, B)$ is equal to the mathematical expectation of the number of all points $(t_1, \dots, t_k) \in D$ whose coordinates t_i, $i = 1, \dots, k$, are entrances of ξ_t into B from $\overline{A}$.

Definition 3. The measure $M_{(k),A}$ will be called the k-leading measure of the flow of entrances into B from $\overline{A}$.

Integrating the k-leading measure over appropriate sets in R^k, one can obtain various moments of the random variables $\eta_A(\Delta, B)$ (see (5)). For example, the following holds.

Theorem 2. *The k-th factorial moment of the number of entrances into B from $\overline{A}$ during the time $(0, t)$ is equal to*

$$M_{(k)}(t, B, A) =$$

$$= k! \int_{y_1 \in B} \int_0^t \mu_A(dy_1)\, ds_1 \cdots \int_{y_k \in B} \int_0^{t - s_1 - \dots - s_{k-1}} M_{y_{k-1},A}(ds_k, dy_k). \qquad (6)$$

Proof follows from the fact that the number of selections $(t_1, \dots, t_k)$, $t_1 < t_2 < \dots < t_k$, from $\eta_A((0,t), B) = \eta$ entrances into B from $\overline{A}$ is equal to $\eta(\eta-1) \cdots (\eta-k+1)/k!$

Studying the asymptotics of integrals of k-leading measures of flows of entrances into B from $\overline{A}$, one can obtain limit theorems on the convergence of "thinned" flows to Poisson flows.

Definition 4. We shall say that a random flow $\eta_\varepsilon(\Delta)$, $\varepsilon > 0$, *converges in distribution* to a flow $\eta_0(\Delta)$ as $\varepsilon \to 0$ ($\eta_\varepsilon(\Delta) \Rightarrow \eta_0(\Delta)$), if for any collection of intervals $\Delta_1, \dots, \Delta_m$,

$$\lim_{\varepsilon \to 0} P_\varepsilon\{\eta_\varepsilon(\Delta_i) = k_i,\ i = 1, \dots, m\} = P_0\{\eta_0(\Delta_i) = k_i,\ i = 1, \dots, m\}.$$

Let $\xi_{t,\varepsilon}$ be a stationary Markov process on $[\mathcal{E}_\varepsilon, \mathfrak{B}_\varepsilon]$, and let $\eta_{A_\varepsilon}(\Delta, B_\varepsilon)$ be the flow of entrances into B_ε from $\overline{A}_\varepsilon$, $\varepsilon > 0$ a parameter, and $M_{x,A_\varepsilon}(\Delta, B_\varepsilon)$ the main function of the flow $\eta_{A_\varepsilon}(\Delta, B_\varepsilon)$. If $\Delta = (t_1, t_2)$, then put

$$\Delta / \mu_{A_\varepsilon}(B_\varepsilon) = (t_1/\mu_{A_\varepsilon}(B_\varepsilon),\ t_2/\mu_{A_\varepsilon}(B_\varepsilon)).$$

Introduce the majorant

$$\overline{M}_{A_\varepsilon}(B_\varepsilon) = \sup_{s>0} \sup_{y \in B_\varepsilon} M_{y,A_\varepsilon}(\Delta_0 + s, B_\varepsilon). \tag{7}$$

Theorem 3. If the fundamental functions $M_{y,A_\varepsilon}(\Delta, B_\varepsilon)$ satisfy the conditions

$$\lim_{\varepsilon \to 0} \overline{M}_{A_\varepsilon}(B_\varepsilon) = 0$$

and, uniformly in $y \in B_\varepsilon$ and $\varepsilon \leq \varepsilon_0$,

$$\lim_{s \to \infty} M_{y,A_\varepsilon}(\Delta_0 + s, B_\varepsilon)/\mu_{A_\varepsilon}(B_\varepsilon) = 1, \tag{9}$$

then

$$\eta_{A_\varepsilon}(\Delta/\mu_{A_\varepsilon}(B_\varepsilon), B_\varepsilon) \Rightarrow \eta_0(\Delta), \tag{10}$$

where $\eta_0(\Delta)$ is a Poisson flow with unit intensity.

The **proof** is based on verifying the convergence of the moments of the random variables $\eta_{A_\varepsilon}(\Delta_i/\mu_{A_\varepsilon}(B_\varepsilon), B_\varepsilon)$, $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$, $i, j = 1, \dots, k$, $k = 1, 2, \dots$, to the moments of mutually independent Poisson random variables with means equal to $|\Delta_i|$. For example, for $k = 1$ it is first proved that asymptotically the integral (6), with A, B, t replaced by $A_\varepsilon, B_\varepsilon, t_\varepsilon = t/\mu_{A_\varepsilon}(B_\varepsilon)$, is equivalent to the integral

$$\begin{aligned} M_{(k)}(\tau_\varepsilon, t_\varepsilon, B_\varepsilon, A_\varepsilon) = k! \int_{y_1 \in B_\varepsilon} \int_0^{t_\varepsilon - (k-1)\tau_\varepsilon} \mu_{A_\varepsilon}(dy_1)\, ds_1 \times \\ \times \int_{y_2 \in B_\varepsilon} \int_{\tau_\varepsilon}^{t_\varepsilon - s_1 - (k-2)\tau_\varepsilon} M_{y_1,A_\varepsilon}(ds_2, dy_2) \cdots \int_{y_k \in B_\varepsilon} \int_{\tau_\varepsilon}^{t - s_1 - \cdots - s_{k-1}} M_{y_{k-1},A_\varepsilon}(ds_k, dy_k) \end{aligned}$$

for a corresponding growth of τ_ε. From (8) we find that, uniformly in $y \in B_\varepsilon$,

$$\lim_{\varepsilon \to 0,\ \tau_\varepsilon \to \infty,\ u_\varepsilon \to \infty} \int_{\tau_\varepsilon}^{\tau_\varepsilon + u_\varepsilon} (\tau_\varepsilon + u_\varepsilon - s)^k M_{y,A_\varepsilon}(ds, B_\varepsilon)/\mu_{A_\varepsilon}(B_\varepsilon)(u_\varepsilon^{k+1}/k+1) = 1.$$

Hence, in turn, we obtain that

$$\lim_{\varepsilon \to 0} M(\tau_\varepsilon, t_\varepsilon, B_\varepsilon, A_\varepsilon) = t^k.$$

The convergence of the finite-dimensional distributions of the quantities $\eta_{A_\varepsilon}(\Delta_i/\mu_{A_\varepsilon}(B_\varepsilon), B_\varepsilon)$ to Poisson distributions is obtained as a consequence of the theorem on convergence of moments [6].

Definition 5. An entrance t into B from $\overline{A}$ will be called a τ-entrance ($\bar{\tau}$-entrance) if in the interval $(t-\tau, t)$ there are no entrances (there are entrances) into B from $\overline{A}$. A τ-entrance into B from $\overline{A}$ will be called the beginning of a τ-packet if it is followed by a $\bar{\tau}$-entrance into B from $\overline{A}$.

Suppose that the original flow is a superposition of m renewal flows having renewal intensities (1), $h(t) \to h_0$, $t \to \infty$. Then the subflow of initial moments of ε-packets, as $\varepsilon \downarrow 0$, converges in distribution to a Poisson flow in a time scale normalized by the intensity of occurrence of ε-packets. This result can be obtained by verifying conditions (8), (9) for the flow of entrances of the Markov process $\xi_t = (u_{1t}, \dots, u_{mt})$, where u_{it} are the backward recurrence times (1) into the set

$$A = \{(x_1, \dots, x_m),\ x_i = 0,\ x_j < \tau,\ x_k > 0,\ i \neq j \neq k,\ i, j, k = 1, \dots, m\}$$

from $\overline{A}$.

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Received
20 XII 1968

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Note: Figure translations are in progress. See original paper for figures.