ON THE MAXIMAL RESOLVABILITY OF PRODUCTS OF TOPOLOGICAL SPACES

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Abstract

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ON THE MAXIMAL RESOLVABILITY OF PRODUCTS OF TOPOLOGICAL SPACES

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In (^4) the maximal resolvability of a product (for three topologies on it) of maximally resolvable spaces was proved, and the question was posed of the validity of this fact for other natural topologies on the product. In this note a positive answer to this question is given (Theorem 2), and propositions are proved which generalize a number of theorems from (^4).

Let $\mathcal{M}$ be an arbitrary family of sets. The cardinal number

$$\Delta\mathcal{M} = \min\{|M| : M \neq \Lambda,\ M \in \mathcal{M}\}$$

will be called the dispersion character of the family $\mathcal{M}$. Recall that the dispersion character ΔX of a space X is the dispersion character of the topology of this space. It is easy to see that for every space* X either $\Delta X = 1$, or $\Delta X \geq \aleph_0$. We shall call a space X k-resolvable if X contains k pairwise disjoint dense subsets of it ($k \geq 1$). In the case of infinite k, this definition is equivalent to the definition of k-resolvability given by Ceder in (^3). A space X is called maximally resolvable (Ceder) if X can be represented as the union of ΔX pairwise disjoint sets, each of which meets every nonempty open subset of X in no fewer than ΔX points. In formulating theorems on maximal resolvability of products it is very convenient that, according to this definition, a space with an isolated point turns out (in a trivial way) to be maximally resolvable. It is easy to see that a space X is maximally resolvable if and only if it is ΔX-resolvable.

V. I. Ponomarev in (^2) gave the definition of a family of sets dense in a space. Let us recall it: a family S of subsets of a space X is dense in X if for every nonempty open set U in X there is a nonempty $s \in S$: $s \subseteq U$.

Theorem 1. *If S is a family of sets dense in X, $|S| \leq \Delta S$, then X is ΔS-resolvable.*

Proof. It is easy to prove that under the hypotheses of the theorem either $\Delta S = 1$, or $\Delta S \geq \aleph_0$. In the first case the assertion of the theorem is trivial. Consider the second case. If $|S| < \Delta S$, then we augment our family by ΔS copies of some one of its elements. In short, we may assume that $|S| = \Delta S$. Let

$S = \{s_\alpha : \alpha \in A\}$. By the Kuratowski lemma (see (^1), Lemma 1) there exists a disjoint family $S' = \{s'_\alpha : \alpha \in A\}$: $s'_\alpha \subseteq s_\alpha$ and $|s'_\alpha| = \Delta S$ for every $\alpha \in A$. Completely order each $s'_\alpha \in S'$: $s'_\alpha = \{x_{\alpha\beta} : \beta < \omega(\Delta S)\}$.[**] Now for each $\beta < \omega(\Delta S)$ put $M_\beta = \{x_{\alpha\beta} : \alpha \in A\}$. It is easy to see that $\{M_\beta : \beta < \omega(\Delta S)\}$ is the required family of ΔS pairwise disjoint dense subsets of X.

Corollary 1. 1°. *If S is a family of sets dense in X, $|S| \le \Delta S$, $\Delta X \le \Delta S$, then X is maximally resolvable.*

2°. If[*] $\pi X \le \Delta X$, then X is maximally resolvable.

3°. ((3), Theorem 3). If $wX \le \Delta X$, then X is maximally resolvable.

Let $\{X_\alpha : \alpha \in A\}$ be an arbitrary family of spaces and let I be a nonempty finitely additive ($B_1, B_2 \in I \Rightarrow B_1 \cup B_2 \in I$) family of subsets of the set A. Then all sets of the form

$$\{x \in \prod_{\alpha \in A} X_\alpha : \ x_\alpha \in U_\alpha, \text{ if } \alpha \in B\},$$

where U_α is open in X_α and $B \in I$, form a base of a certain topology in $\prod_{\alpha \in A} X_\alpha$. If $B' \subseteq B \in I$, then the set

$$\{x \in \prod_{\alpha \in A} X_\alpha : \ x_\alpha \in U_\alpha, \text{ if } \alpha \in B'\},$$

where U_α is open in X_α, is open in this topology. Therefore one may assume at once that I is an ideal. This topology is a T_0-topology if and only if I covers A. Thus, let I be an ideal of its subsets covering A. The space consisting of the set

$$\prod_{\alpha \in A} X_\alpha$$

and the topology defined above will be denoted by

$$\prod_{\alpha \in A}^{I} X_\alpha.$$

Lemma 1. If $A \notin I$, then in the space $\prod_{\alpha \in A}^{I} X_\alpha$ the family

$$S : \ |S| = \sum_{B \in I} \prod_{\alpha \in B} sX_\alpha, \qquad \Delta S = \prod_{\alpha \in A \setminus B_0} |X_\alpha|,$$

is dense, for some $B_0 \in I$.

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[*] By a space we shall mean a T_0-space.

[**] $\omega(\tau)$ is the initial ordinal number of cardinality τ.

If $A \notin I$, put

$$J = \{B \in I : \prod_{\alpha \in A \setminus B} |X_\alpha| = \min_{B \in I} \prod_{\alpha \in A \setminus B} |X_\alpha|\}.$$

Of course, $J \neq \Lambda$. By $I|C$, where $C \subseteq A$, we shall denote the family

$$\{B \cap C : \ B \in I\} = \{B \in I : \ B \subseteq C\}.$$

$I|C$ is a subideal of the ideal I.

Lemma 2. Let $\{X_\alpha : \alpha \in A\}$ be an infinite family of non-one-point spaces, and let I be an ideal of its subsets covering A, $I \neq A$. Let $B \in J$ and

$$X = \prod_{\alpha \in A \setminus B}^{I_0} X_\alpha,$$

where $I_0 = I|\,A \setminus B$. Then: 1) $\Delta X = |X| \geq c = 2^{\aleph_0}$; 2) X is maximally resolvable.

Theorem 2. If $\{X_\alpha : \alpha \in A\}$ is a family of maximally resolvable spaces and I is an ideal of its subsets covering A, then

$$\prod_{\alpha \in A}^{I} X_\alpha$$

is maximally resolvable.

Theorem 2 contains, as special cases, Theorems 2 and 9 of (4), and gives an exhaustive answer to the question of Cedar and Pearson posed by them in (4) (see the end of § 3).

Theorem 3. Let $\{X_\alpha : \alpha \in A\}$ be an infinite family of non-one-point spaces, and let I be an ideal of its subsets covering A, $I \neq A$. Then

$$X = \prod_{\alpha \in A}^{I_0} X_\alpha$$

is c-resolvable.

Theorem 3 is a generalization of Theorem 12 of (4), and in the case of infinite products, a substantial strengthening of Hewitt' s theorem (see (5), Theorem 45).

Let now I be the ideal of finite subsets of A. Then the topology of the space

$$X = \prod_{\alpha \in A}^{I} X_\alpha$$

is the Tikhonov topology, and we shall, as usual, write

$$X = \prod_{\alpha \in A} X_\alpha.$$

If $\Delta X \geq \Delta Y$, then we shall say that X majorizes Y (X dominates Y according to (4)). In (4) it is proved (Lemma 2) that if X is maximally resolvable and majorizes Y, then $X \times Y$ is maximally resolvable. Hence (and from the fact that X majorizes $\prod_{i=1}^{k} Y_i$ if it majorizes Y_i, $1 \leq i \leq k$) it follows immediately:

Assertion 1. *If one of the spaces* $X_1, \dots, X_k$ *is maximally resolvable and majorizes each of the others, then*

$$\prod_{i=1}^{k} X_i$$

is maximally resolvable.

For the Tychonoff topology, Lemma 2 can be given a more convenient formulation. Let $R = \{\tau_\alpha : \alpha \in A\}$ be a family of cardinal numbers. We shall call a cardinal number τ a limit point of the family R if every neighborhood of it (in the order topology) contains infinitely many elements of R. By the upper limit $\overline{\lim}\, R$ of the family R we shall mean the least upper bound of the set of all limit points of R. Obviously, R has an upper limit if and only if it is infinite. Put

$$\sup R = \sup_{\alpha \in A} \tau_\alpha$$

and

$$A(R) = \{\alpha \in A : \tau_\alpha > \overline{\lim}\, R\}.$$

Clearly, $A(R)$ is finite.

Lemma 2^T**.** *Let* $\{X_\alpha : \alpha \in A\}$ *be an infinite family of spaces and*

$$R = \{\tau_\alpha = [\Delta X_\alpha, |X_\alpha|] : \alpha \in A\}.$$

Then

$$X = \prod_{\alpha \in A/A(R)} X_\alpha$$

is maximally resolvable, and moreover

$$\Delta X \geq \overline{\lim}\, R.$$

Now Assertion 1 can be given the following final form:

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* πX –the π-weight of the space (V. I. Ponomarev) –is $d\mathcal{M}$, where $\mathcal{M}$ is the family of all families of open subsets of X dense in X.

Theorem 4. *If* $\{X_\alpha : \alpha \in A\}$ *is a family of spaces, one of which is maximally resolvable and majorizes each of the others, then*

$$X = \prod_{\alpha \in A} X_\alpha$$

is maximally resolvable.

We note that this theorem implies Theorem 6 of (^{4}).

Denote by $\|\mathcal{M}\|$ the least upper bound of the pointwise cardinalities of the family $\mathcal{M}$ of subsets of some set.

Lemma 3 (see Proposition 2 and the proof of Theorem 4 in (^{4})). *Let* $\mathcal{M}$ *be a family of subsets dense in* Y*. If* X *is* $\|\mathcal{M}\|$*-resolvable, then* $X \times Y$ *is* $|\mathcal{M}|$*-resolvable.*

Proof. Let $\mathcal{M} = \{M_\lambda : \lambda \in \mathcal{L}\}$. To each point $y \in Y$ we assign the set

$$\mathcal{L}_y = \{\lambda \in \mathcal{L} : y \in M_\lambda\}.$$

Since $|\mathcal{L}_y| \leq \|\mathcal{M}\|$, there exists a one-to-one mapping of $\mathcal{L}_y$ into a system $\{A\}$ consisting of $\|\mathcal{M}\|$ pairwise disjoint dense subsets of X. The image of an element $\lambda \in \mathcal{L}_y$ under this mapping will be denoted by A^y_λ. Put

$$B_\lambda = \bigcup_{y \in M_\lambda} (A^y_\lambda \times \{y\}).$$

It is easy to see that $\{B_\lambda : \lambda \in \mathcal{L}\}$ is a family consisting of $|\mathcal{M}|$ pairwise disjoint dense subsets of $X \times Y$.

Denote by γX the dispersion character of the family of all open dense subsets of X. Obviously,

$$\Delta X \leq \gamma X \leq |X|$$

and

$$\gamma \prod_{i=1}^{n} X_i = \max_i \gamma X_i$$

if at least one γX_i is infinite. The following theorem slightly strengthens Theorem 4 of (^{4}).

Theorem 5. *If* X *is maximally resolvable and* $*\gamma Y \leq \Delta X^+$*, then* $X \times Y$ *is maximally resolvable.*

Proof. It is evidently necessary to prove only the case

$$\Delta Y = \gamma Y = \Delta X^+.$$

Let M be an open dense subset of Y of cardinality ΔY. Well-order it:

$$M = \{y_\alpha : \alpha < \omega(\Delta Y)\}.$$

Put

$$M_\beta = \{y_\alpha \in M : \beta \le \alpha < \omega(\Delta Y)\}.$$

It is easy to see that

$$\mathcal{M} = \{M_\beta : \beta < \omega(\Delta Y)\}$$

is a family of dense subsets of Y, $\|\mathcal{M}\| = \Delta X$, $|\mathcal{M}| = \Delta Y = \Delta(X \times Y)$. The assertion of the theorem now follows from Lemma 3.

$*$ τ^+ is the first cardinal number greater than τ.

Corollary 2. Let $X_0, X_1, \dots, X_n$ be a finite family of spaces and let X_0 be maximally resolvable. Suppose the following condition holds: if

$$\Delta X_0 < \tau \le \max_{1 \le i \le n} \Delta X_i,$$

then there exists i, $1 \le i \le n$, such that $\Delta X_i = \gamma X_i = \tau$. Then

$$\prod_{i=0}^{n} X_i$$

is maximally resolvable.

Proof. From our family one can select a subfamily $X_{i_0}, X_{i_1}, \dots, X_{i_k}$, where $i_0 = 0$ and $\Delta X_{i_k} = \max_{1 \le i \le n} \Delta X_i$, such that

$$\Delta X_{i_{s+1}} = \gamma X_{i_{s+1}} = \Delta X_{i_s}^+, \quad s = 0, 1, \dots, k-1.$$

Applying Theorem 5 successively, we obtain that

$$\prod_{s=0}^{k} X_{i_s}$$

is maximally resolvable. It remains only to note that this product majorizes each of the spaces X_i, $i = 0, 1, \dots, n$, and to apply Assertion 1.

Theorem 6. Let $\{X_\alpha : \alpha \in A\}$ be an infinite family of spaces. Suppose that for each $\alpha \in A$ one can choose τ_α from the interval $[\Delta X_\alpha, |X_\alpha|]$ in such a way that the following condition holds: if

$$\overline{\lim}\, R < \tau \le \sup R,$$

where $R = \{\tau_\alpha : \alpha \in A\}$, then there exists $\alpha \in A$ such that

$$\tau_\alpha = \Delta X_\alpha = \gamma X_\alpha = \tau.$$

Then

$$X = \prod_{\alpha \in A} X_\alpha$$

is maximally resolvable.

Proof. By Lemma 2^τ,

$$X_0 = \prod_{\alpha \in A \setminus A(R)} X_\alpha$$

is maximally resolvable and

$$\Delta X_0 \geq \overline{\lim}\, R.$$

It remains only to apply Corollary 2 to the family $\{X_0; X_\alpha : \alpha \in A(R)\}$.

This theorem substantially strengthens Theorem 5 of (4) (for the case of an infinite product).

Corollary 3 ((4), Theorem 3). Let $\{X_\alpha : \alpha \in A\}$ be a family of spaces. If

$$\{\alpha \in A : \ |X_\alpha| = \sup_{\alpha \in A} |X_\alpha|\}$$

is empty or infinite, then

$$X = \prod_{\alpha \in A} X_\alpha$$

is maximally resolvable.

Proof. Let

$$R = \{|X_\alpha| : \alpha \in A\}.$$

Then

$$\overline{\lim}\, R = \sup R,$$

so that we trivially find ourselves in the hypotheses of Theorem 6.

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Note: Figure translations are in progress. See original paper for figures.