COMPACT APPROXIMATION OF OPERATORS AND APPROXIMATE SOLUTION OF OPERATOR EQUATIONS

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COMPACT APPROXIMATION OF OPERATORS AND APPROXIMATE SOLUTION OF OPERATOR EQUATIONS

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The concept of compact approximation of operators was introduced in ([1,2]). The situation considered below is more general in several respects.

1. Let E, F, E_n, F_n $(n = 1, 2, \ldots)$ be Banach spaces. Suppose there exist operators $*$ $p_n \in \mathcal{L}(E, E_n)$, $q_n \in \mathcal{L}(F, F_n)$ with the following properties:

$$p_n E = E_n, \qquad q_n F = F_n \qquad (n = 1, 2, \ldots); \tag{1}$$

$$\|p_n\| \leq c, \qquad \|q_n\| \leq c \qquad (n = 1, 2, \ldots;\ c = \text{const}); \tag{2}$$

$$\|\xi_n\|_{E_n} \geq \gamma \inf_{x \in E,\, p_{nx} = \xi_n} \|x\|_E, \qquad \|\eta_n\|_{F_n} \geq \gamma \inf_{y \in F,\, q_{ny} = \eta_n} \|y\|_F \tag{3}$$

for any $**$ $\xi_n \in E_n,\ \eta_n \in F_n$ $(n = 1, 2, \ldots;\ \gamma = \text{const} > 0)$;

$$\lim_{n \to \infty} \|p_{nx}\| > 0, \qquad \lim_{n \to \infty} \|q_{ny}\| > 0 \quad \text{for } \forall x \in E,\ y \in F\ (x \neq 0,\ y \neq 0). \tag{4}$$

Definition 1. A sequence of operators $T_n \in \mathcal{L}(E_n, F_n)$ **compactly approximates** an operator $T \in \mathcal{L}(E, F)$ with respect to $\{p_n\}$ and $\{q_n\}$, if the following conditions are satisfied:

a) $\|q_{nTx} - T_{np_{nx}}\| \to 0$ as $n \to \infty$ for every $x \in E$;

b) for any sequence $\{\xi_n\}$ $(\xi_n \in E_n,\ \|\xi_n\| \leq 1,\ n = 1, 2, \ldots)$ there exist such $x_n \in E,\ y_n \in F$ and $c' = \text{const}$ that $p_{nx}n = \xi_n,\ q_{ny}n = T_n\xi_n,\ \|y_n\| \leq c'\|T_n\xi_n\|$ $(n = 1, 2, \ldots)$, and the sequence $\{y_n - Tx_n\}$ is compact in F.

Remark 1. If $T \in \mathcal{L}(E, F)$ is completely continuous, then condition b) is equivalent to the following condition (cf. ([1])): for any sequence $\{\xi_n\}$ ($\xi_n \in E_n$, $\|\xi_n\| \le 1$, $n = 1, 2, ...$) there exist such $y_n \in F$ that $q_{ny}n = T_n \xi_n$ ($n = 1, 2, ...$), and the sequence $\{y_n\}$ is compact in F.

Consider the equations

$$Tx = y \qquad (y \in F), \tag{5}$$

$$T_n \xi_n = q_{ny}. \tag{6}$$

Theorem 1. *Suppose the following conditions are satisfied:*

1) *the sequence $T_n \in \mathcal{L}(E_n, F_n)$ compactly approximates $T \in \mathcal{L}(E, F)$;*
2) *T has an inverse $T^{-1} \in \mathcal{L}(F, E)$;*
3) *the operators T_n are such that from*

$$\inf_{\xi_n \in E_n,\ \|\xi_n\|=1} \|T_n \xi_n\| > 0$$

it follows that there exists an inverse $T_n^{-1} \in \mathcal{L}(F_n, E_n)$.

Then equation (6) has, for all sufficiently large n, a unique solution $\xi_n^* \in E_n$, and $\|\xi_n^* - p_{nx}^*\| \to 0$ as $n \to \infty$, where $x^* = T^{-1}y$ is the solu-

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* By $\mathcal{L}(X, Y)$ is denoted the space of linear continuous operators from X to Y.

** The formula $\varphi_n \xi_n = p_n^{-1}(\xi_n)$ defines an isomorphism $\varphi_n \in \mathcal{L}(E_n, E/p_n^{-1}(0))$, for which $\|\varphi_n\| \le 1/\gamma$, $\|\varphi_n^{-1}\| \le c$ ($n = 1, 2, ...$); here $p_n^{-1}(\xi_n)$ is the complete preimage of the element $\xi_n \in E_n$. Similarly, F_n is isomorphic to the quotient space $F/q_n^{-1}(0)$.

solution of equation (5). The estimate is valid

$$c_1 \|q_{nTx}^* - T_{np_{nx}}^*\| \le \|\xi_n^* - p_{nx}^*\| \le c_2 \|q_{nTx}^* - T_{np_{nx}}^*\| \qquad (c_1, c_2 = \text{const} > 0). \tag{7}$$

Proof. From 1) it follows that

$$\|T_n\| \le 1/c_1 = \text{const} \qquad (n = 1, 2, ...). \tag{8}$$

Let us show that, for sufficiently large n, there exist $T_n^{-1} \in \mathcal{L}(F_n, E_n)$ and

$$\|T_n^{-1}\| \leq c_2 = \text{const} \qquad (n = n_0, n_0 + 1, ...). \tag{9}$$

Suppose that for some $\xi_n \in E_n$ ($\|\xi_n\| = 1,\ n = 1, 2, ...$) we have $\|T_n \xi_n\| \to 0$ as $n \to \infty$. Choose $x_n \in E,\ y_n \in F$ ($n = 1, 2, ...$) such that $p_{nx} n = \xi_n,\ q_{ny} n = T_n \xi_n,\ \|y_n\| \leq c' \|T_n \xi_n\| \to 0$, and $\{y_n - T x_n\}$ is compact in F. Then $\{x_n\}$ is compact in E, and from condition a) of Definition 1 we conclude that $\|q_{nTx} n - T_{np_{nx}} n\| \to 0$ as $n \to \infty$. Since $\|T_{np_{nx}} n\| = \|T_n \xi_n\| \to 0$, it follows that $\|q_{nTx} n\| \to 0$, and also $\|q'_{nTx}\| \to 0$ as $n \to \infty$ for every limit point x' of the sequence $\{x_n\}$; by (4), $T x' = 0$ and $x' = 0$. Hence $x_n \to 0$ as $n \to \infty$, which contradicts (2) and the equalities $\|p_{nx} n\| = \|\xi_n\| = 1$. Together with 3), this contradiction proves (9).

From inequalities (8), (9) and the equality

$$T_n(\xi_n^* - p_{nx}^*) = q_{nTx}^* - T_{np_{nx}}^*$$

the estimate (7) follows. The convergence $\|\xi_n^* - p_{nx}^*\| \to 0$ follows from (7) and condition a) of Definition 1. Theorem 1 is proved.

Inequalities (9), together with condition a) of Definition 1, mean, in the terminology of ([3],[4]), that the sequence $T_n \in \mathcal{L}(E_n, F_n)$ stably approximates $T \in \mathcal{L}(E, F)$. In ([3],[4]) it is proved that stability of the approximation is not only a sufficient but also a necessary condition for the convergence $\|\xi_n^* - p_{nx}^*\| \to 0$ for any right-hand side $y \in F$ of equation (5). Theorem 1 can be reformulated as follows: if the sequence $T_n \in \mathcal{L}(E_n, F_n)$ compactly approximates $T \in \mathcal{L}(E, F)$ and if conditions 2) and 3) of Theorem 1 are fulfilled, then the approximation is stable. The converse assertion is true only in the following weakened form.

Remark 2. Let the sequence $\widetilde{T}_n \in \mathcal{L}(E_n, F_n)$ stably approximate $T \in \mathcal{L}(E, F)$. Let $TE = F$, and let T admit a compact approximation by some sequence of operators $T_n \in \mathcal{L}(E_n, F_n)$, for which condition 3) of Theorem 1 is fulfilled. Then there exist $\tilde{q}_n \in \mathcal{L}(F, F_n)$ with properties (1)–(4) such that, with respect to $\{p_n\}$ and $\{\tilde{q}_n\}$, the sequence $\widetilde{T}_n$ compactly approximates T.

2. Consider the case where the operators $T : E \to F$ and $T_n : E_n \to F_n$ in equations (5) and (6) are nonlinear.

Lemma 1. Let the operator T_n be Fréchet differentiable in the ball $\{\xi_n \in E_n : \|\xi_n - \xi_n^0\| \leq \delta_0\}$, where $\delta_0 > 0,\ \xi_n^0 \in E_n$. Let $T'_n(\xi_n^0) \in \mathcal{L}(E_n, F_n)$ have an inverse $[T'_n(\xi_n^0)]^{-1} \in \mathcal{L}(F_n, E_n)$, and suppose that for some δ and θ ($0 < \delta \leq \delta_0,\ 0 \leq \theta < 1$) the inequalities

$$\sup_{\xi_n \in E_n,\ \|\xi_n - \xi_n^0\| \leq \delta} \|[T'_n(\xi_n^0)]^{-1} [T'_n(\xi_n) - T'_n(\xi_n^0)]\| \leq \theta, \tag{10}$$

$$\alpha_n \equiv \|[T'_n(\xi_n^0)]^{-1} [T_n \xi_n^0 - q_{ny}]\| \leq \delta(1 - \theta). \tag{11}$$

Then equation (6) has, in the ball $\|\xi_n - \xi_n^0\| \le \delta$, a unique solution ξ_n^*, and the estimate is valid

$$\alpha_n/(1+\theta) \le \|\xi_n^* - \xi_n^0\| \le \alpha_n/(1-\theta). \tag{12}$$

The proof is analogous to the proof of Theorem 2 from ([5]).

Theorem 2. Suppose that the following conditions are fulfilled:

1) equation (5) has a solution x^*;
2) the operator T is Fréchet differentiable at the point x^*, and $T'(x^*) \in \mathcal{L}(E,F)$ has an inverse $[T'(x^*)]^{-1} \in \mathcal{L}(F,E)$;
3) the operators T_n $(n = 1, 2, ...)$ are Fréchet differentiable in the corresponding balls $\|\xi_n - p_{nx}^*\| \leqslant \delta_0$ $(\delta_0 = \text{const} > 0)$, and for any $\varepsilon > 0$ there exist n_ε and η_ε $(0 < \eta_\varepsilon \leqslant \delta_0)$ such that $\|T_n'(\xi_n) - T_n'(p_{nx}^*)\| \leqslant \varepsilon$ for $n \geqslant n_\varepsilon$, $\|\xi_n - p_{nx}^*\| \leqslant \eta_\varepsilon$;
4) $\|q_{nTx}^* - T_{np_{nx}}^*\| \to 0$ as $n \to \infty$ $(q_{nTx}^* = q_{ny})$;
5) the sequence $T_n'(p_{nx}^*) \in \mathcal{L}(E_n, F_n)$ compactly approximates the operator $T'(x^*) \in \mathcal{L}(E,F)$;
6) the operators $T_n'(p_{nx}^*)$ are such that from

$$\inf_{\xi_n \in E_n,\ \|\xi_n\|=1} \|T_n'(p_{nx}^*)\xi_n\| > 0$$

there follows the existence of an inverse $[T_n'(p_{nx}^*)]^{-1} \in \mathcal{L}(F_n, E_n)$.

Then there exist N and $\delta > 0$ such that, for $n \geqslant N$, equation (6) has in the ball $\|\xi_n - p_{nx}^*\| \leqslant \delta$ a unique solution ξ_n^*. As $n \to \infty$, the convergence $\|\xi_n^* - p_{nx}^*\| \to 0$ holds with the estimate (7).

Proof is based on Lemma 1, in which we put $\xi_n^0 = p_{nx}^*$. From 2), 5), and 6) we conclude that, for sufficiently large n, the inverses $[T_n'(p_{nx}^*)]^{-1} \in \mathcal{L}(F_n, E_n)$ exist, and their norms are bounded in the aggregate. Fix θ $(0 < \theta < 1)$. Using condition 3), we find $\delta > 0$ such that, for sufficiently large n, (10) will be satisfied. By virtue of 4), for sufficiently large n (11) is also fulfilled. From (12) we obtain the estimate (7); from (7) and 4) follows the convergence $\|\xi_n^* - p_{nx}^*\| \to 0$ as $n \to \infty$. Theorem 2 is proved.

3. Let now $E = F$, $E_n = F_n$, $p_n = q_n$ $(n = 1, 2, ...)$. In studying the closeness of solutions of the operator equations

$$x = Tx, \tag{13}$$

$$\xi_n = T_n \xi_n \tag{14}$$

with completely continuous operators T and T_n, it is natural to use the concept of the rotation of vector fields ([6]). We shall assume that the Banach spaces E and E_n $(n = 1, 2, ...)$ are real. Let Ω be a bounded domain (a connected open set) in E; denote $\Omega_n = p_n\Omega$. Then Ω_n is a bounded domain in E_n. By $\overline{\Omega}$, $\dot{\Omega}$, $\overline{\Omega}_n$, $\dot{\Omega}_n$ we denote the closures and boundaries of the domains Ω and Ω_n. It is clear that $p_n\overline{\Omega} \subset \overline{\Omega}_n$.

Definition 2. A sequence of operators $T_n : \overline{\Omega}_n \to E_n$ **compactly approximates** the completely continuous operator $T : \overline{\Omega} \to E$ with respect to $\{p_n\}$, if the following conditions are fulfilled:

a) $\|p_{nTx} - T_{np_{nx}}\| \to 0$ as $n \to \infty$ for every $x \in \overline{\Omega}$;

b) for any sequence $\{\xi_n\}$ $(\xi_n \in \overline{\Omega}_n,\ n = 1, 2, ...)$ there exist such $y_n \in E$ that $p_{ny}n = T_n\xi_n$ $(n = 1, 2, ...)$ and the sequence $\{y_n\}$ is compact in E.

Below we assume that (4) is fulfilled in the following strengthened form:

$$\lim_{n\to\infty} \|p_{nx}\| \geqslant c_0\|x\| \quad \text{for all } x \in E \ (c_0 = \text{const} > 0). \tag{4'}$$

Lemma 2. Suppose the following conditions are fulfilled:

1) the sequence of completely continuous operators $T_n : \overline{\Omega}_n \to E_n$ compactly approximates the completely continuous operator $T : \overline{\Omega} \to E$;

2) the operators T_n $(n = 1, 2, ...)$ are such that from $\|\xi_n - p_{nx}\| \to 0$ $(\xi_n \to \overline{\Omega}_n,\ x \in \overline{\Omega})$ it follows that $\|T_n\xi_n - T_{np_{nx}}\| \to 0$ as $n \to \infty$;

3)

$$\lim_{n\to\infty} \inf_{\xi_n \in \dot{\Omega}_n} \|p_{nx} - \xi_n\| > 0 \quad \text{for every } x \notin \dot{\Omega} \ (x \in E);$$

4) the operator T has no fixed points on the boundary $\dot{\Omega}$.

Then, for all sufficiently large n, the operator T_n has no fixed points on the boundary $\dot{\Omega}_n$, and the equality of rotations holds:

$$\gamma(\xi_n - T_n\xi_n; \dot{\Omega}_n) = \gamma(x - Tx; \dot{\Omega}) \quad (n = n_0, n_0 + 1, ...).$$

Remark 3. Condition 3) of Lemma 2 is fulfilled if (2) and (4′) hold with $c = c_0 = 1$, and Ω is a ball in E.

Theorem 3. Suppose conditions 1)–3) of Lemma 2 are satisfied, and suppose equation (13) has an isolated solution $x^* \in \Omega$ of nonzero index (6), unique in $\overline{\Omega}$. Then equation (14), for sufficiently large n, has at least one solution $\xi_n^* \in \overline{\Omega}_n$, and $\|\xi_n^* - p_nx^*\| \to 0$ as $n \to \infty$ for all solutions $\xi_n^* \in \overline{\Omega}_n$.

4. As an application, we consider the method of mechanical quadratures for solving integral equations. Let D be a metric compact set, ρ a metric in D, and ν a measure defined on some algebra of subsets of D. Assume that $|\nu|(D) < \infty$ (where $|\nu|$ is the total variation of ν) and that the ball $S(t_0, r) = \{t \in D : \rho(t, t_0) < r\}$ is ν-measurable for any $t_0 \in D$, $r > 0$, with $|\nu|(S(t_0, r)) > 0$. Clearly, every function continuous on D is ν-integrable. Consider the integral equation

$$x(t) = \int_D K(t, s, x(s))\, \nu(ds), \tag{15}$$

a convergent quadrature process

$$\int_D z(s)\, \nu(ds) = \sum_{j=1}^{n} \alpha_{jn} z(s_{jn}) + \Phi_n(z) \qquad (n = 1, 2, \dots)$$

$$(\Phi_n(z) \to 0 \text{ for continuous functions } z(s))$$

and the system of equations

$$\xi_{in} = \sum_{j=1}^{n} \alpha_{jn} K(s_{in}, s_{jn}, \xi_{jn}) \qquad (i = 1, \dots, n). \tag{16}$$

Theorem 4. Suppose equation (15) has a solution $x^*(t)$ (continuous on D), and suppose the kernel $K(t, s, x)$ is continuous and has a continuous derivative $\partial K(t, s, x)/\partial x$ for $t, s \in D$, $|x - x^*(s)| \le \delta_0$ ($\delta_0 = \text{const} > 0$). Suppose the equation

$$y(t) = \int_D H(t, s) y(s) \nu(ds) \qquad (H(t, s) = \partial K(t, s, x^*(s))/\partial x)$$

has only the zero solution. Then there exist N and $\delta > 0$ such that, for $n \ge N$, the system of equations (16) has a unique solution $(\xi^*_{1n}, \dots, \xi^*_{nn})$ such that $|\xi^*_{jn} - x^*(s_{jn})| \le \delta$ $(j = 1, \dots, n)$. The convergence

$$\max_{1 \le j \le n} |\xi^*_{jn} - x^*(s_{jn})| \to 0 \quad \text{as } n \to \infty \tag{17}$$

holds, with the estimate

$$c_1 \varepsilon_n \le \max_{1 \le j \le n} |\xi^*_{jn} - x^*(s_{jn})| \le c_2 \varepsilon_n \qquad (c_1, c_2 = \text{const} > 0),$$

where

$$\varepsilon_n = \max_{1\le i\le n} |\Phi_n(z_{in})|, \qquad z_{in}(s) = K(s_{in}, s, x^*(s)).$$

The proof is based on Theorem 2, which is applied with $E = F = C(D)$, $E_n = F_n = m_n$, $p_{nx} = q_{nx} = (x(s_{1n}), \dots, x(s_{nn}))$.

Theorem 3 makes it possible to establish the convergence (17), assuming only the continuity of the kernel $K(t, s, x)$ and the existence of a solution $x^*(t)$ of nonzero index (6) of equation (15).

Similar results in the case of Lebesgue measure on $D = [a, b] \subset R^1$ were obtained by another method in [5], and for linear equations in a case similar to the one considered, in [1].

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Note: Figure translations are in progress. See original paper for figures.