MOTION OF A MEDIUM WITH ULTRARELATIVISTIC VELOCITIES IN THE GENERAL THEORY OF RELATIVITY

PHYSICS

1969

SovietRxiv

Abstract

Full Text

UDC 530-12

PHYSICS

K. P. STANYUKOVICH

MOTION OF A MEDIUM WITH ULTRARELATIVISTIC VELOCITIES IN THE GENERAL THEORY OF RELATIVITY

(Presented by Academician Ya. B. Zel’ dovich, 9 X 1968)

The basic equations describing the motion of a medium in a centrally symmetric field in the general theory of relativity, and the field itself, will be written in the form[1]

$$\frac{1}{c^2\vartheta^2}\left[Au_t + uu_r\right] - \frac{\omega^2}{c^2}\left[(\ln v)_r - \frac{Au}{c^2}(\ln v)_t\right] = \frac{1}{2u}\left[A\lambda_t + u\lambda_r\right] + \frac{\theta^2 T^0 \sigma_r}{W}; \quad (1)$$

$$-[A(\ln v)_t + u(\ln v)_r] + \frac{1}{\theta^2}\left[u_r + \frac{Auu_t}{c^2}\right] + \frac{2u}{r} = \frac{u}{2}\left[\lambda_r + \frac{Au\lambda_t}{c^2}\right]; \quad (2)$$

$$A\sigma_t + u\sigma_r = 0. \quad (3)$$

Here $A = e^{(\lambda-\nu)/2}$; $u = A\,dr/dt$; $\theta^2 = 1 - u^2/c^2$; $W = (p+\varepsilon)v$; u is the 3-velocity; p is the pressure; v is the specific volume; $\varepsilon = \rho c^2$ is the energy density; W is the heat content, $\omega^2/c^2 = -(\partial \ln W/\partial \ln v)_\sigma$; ω is the speed of sound; σ is the entropy; T^0 is the temperature. In addition one must know the equation of state of the medium $p = p(\sigma; v)$ and use the identity $\partial(p; v)/\partial(T; \sigma) = 1$.

We shall write the two independent field equations in the form:

$$(re^{-\lambda})_r = 1 - \frac{\chi r^2}{\theta^2}\left[\varepsilon + pu^2/c^2\right]; \quad (4)$$

$$A(re^{-\lambda})_t = \frac{\chi u r^2}{\theta^2}\left[\varepsilon + p\right]. \quad (5)$$

Let us consider the motion when $u/c = 1 - 2\Delta$, where $\Delta \ll 1$,

---

[1]

$$p = \sigma v^{-k} = (k-1)\varepsilon; \qquad \omega^2/c^2 = (k-1); \tag{6}$$

neglecting terms of order Δ^2, we arrive at the system of equations

$$A x_\tau + x_r = 0, \tag{7}$$

where $x = \ln[p\, r^{2k/(2-k)}]$, $\tau = ct$;

$$A y_\tau + y_r = 0, \tag{8}$$

where $y = \ln[\Delta e^{\lambda} r^{4(k-1)/(2-k)}]$;

$$A \sigma_\tau + \sigma_r = 0; \tag{9}$$

$$\lambda_r = \frac{1}{r} + e^{\lambda}\left[\frac{\chi r}{4\Delta}(\varepsilon + p) - \left(\frac{1}{r} + \chi r p\right)\right]; \tag{10}$$

$$A \lambda_\tau = -\frac{\chi r^2}{4\Delta} e^{\lambda} (\varepsilon + p). \tag{11}$$

We introduce a new independent variable m by means of the relation

$$\chi\left(\frac{\partial m}{\partial r}\right)_t = \frac{\chi r^2}{\theta^2}\left(\varepsilon + \frac{p u^2}{c^2}\right) = 1 - (r e^{-\lambda})_r,$$

whence

$$m = \int_0^r \frac{r^2}{\theta^2}\left(\varepsilon + p\frac{u^2}{c^2}\right) dr = \frac{r}{\chi}(1 - e^{-\lambda});$$

for $\Delta \ll 1$ these relations take the form

$$\varkappa\left(\frac{\partial m}{\partial t}\right)_r = \frac{\varkappa r^2}{4\Delta}(\varepsilon + p) - \varkappa r^2 p = 1 - (r e^{-\lambda})_r, \tag{12}$$

$$m = \int_0^r \left[\frac{r^2}{4\Delta}(\varepsilon + p) - r^2 p\right] dr = \frac{\varkappa r}{\varkappa}(1 - e^{-\lambda}).$$

Let us now pass to the independent variables $(m; r)$. The system of equations (10), (11) then takes the form

$$r\frac{\partial\lambda}{\partial m}\frac{\partial\tau}{\partial r}+\frac{\partial\tau}{\partial m}\left[1-e^{-\lambda}(1+\varkappa r^2 p)+\frac{\varkappa r^2}{4\Delta}e^{\lambda}(\varepsilon+p)-r\frac{\partial\lambda}{\partial r}\right]=0,$$

$$A\frac{\partial\lambda}{\partial m}+\frac{\varkappa r e^{\lambda}}{4\Delta}(\varepsilon+p)\frac{\partial\tau}{\partial m}=0.$$

Since $\lambda=-\ln(1-\varkappa m/r)$, $\partial\lambda/\partial m=\varkappa e^{\lambda}/r$; $\partial\lambda/\partial r=-\varkappa m e^{\lambda}/r^2$, from these relations we find

$$\partial\tau/\partial m=-4\Delta A/r^2(p+\varepsilon),\qquad \partial\tau/\partial r=A[1-4\Delta p/(p+\varepsilon)]. \tag{13}$$

Equations (7), (8), (9) in the variables $(m;r)$ may be written in the form

$$r^2 p x_m=x_r; \tag{14}$$

$$r^2 p y_m=y_r; \tag{15}$$

$$r^2 p\sigma_m=\sigma_r. \tag{16}$$

Write equation (14) in the form

$$e^{x}r^{4(1-k)/(2-k)}\partial x/\partial m=\partial x/\partial r. \tag{17}$$

Its general solution has the form

$$m=\frac{(2-k)}{(5k-6)}e^{x}r^{(6-5k)/(2-k)}+F_1(x)=\frac{2-k}{5k-6}pr^3+F_1(pr^{2k/(2-k)}). \tag{18}$$

Writing the system (14), (15), and (16) in the independent variables $(x;r)$, we find that $y=y(x)$, $\sigma=\sigma(x)$, and, consequently,

$$m=\frac{2-k}{5k-6}pr^3+F_2(y)=\frac{2-k}{5k-6}pr^3+F_3(\sigma). \tag{19}$$

At the same time

$$\Delta=r^{-4(k-1)/(2-k)}(1-\varkappa m/r)f_1(pr^{2k/(2-k)}); \tag{20}$$

$$\sigma=f_3(pr^{2k/(2-k)}). \tag{21}$$

On the basis of equation (12) we have

$$\left(\frac{\partial m}{\partial r}\right)_\tau = \frac{r^2(\varepsilon+p)}{4\Delta} - pr^2 = \frac{k}{4(k-1)}\frac{pr^2}{\Delta} - pr^2 = \frac{k}{4(k-1)}e^{x-y+\lambda} - pr^2 \quad (22)$$

or

$$(\partial m/\partial r)_\tau = f(x)/(1-\varkappa m)r - pr^2, \quad (23)$$

where

$$f(x) = ke^{x-y}/4(k+1); \qquad r^2(\varepsilon+p)/4\Delta = f(x)e^\lambda = f(x)/(1-\varkappa m/r).$$

From (18) we find that

$$\left(\frac{\partial m}{\partial r}\right)_\tau = -pr^2 + \left(\frac{\partial x}{\partial r}\right)_\tau \left[\frac{2-k}{5k-6}e^x r^{(6-5k)/(2-k)} + \frac{dF_1(x)}{dx}\right]. \quad (24)$$

Comparing (23) and (24), we arrive at the result:

$$\left(\frac{\partial x}{\partial r}\right)_\tau = \frac{f(x)}{\left(1-\frac{\varkappa m}{r}\right)\left(\frac{2-k}{5k-6}e^x r^{(6-5k)/(2-k)} + \frac{dF_1(x)}{dx}\right)}, \quad (25)$$

where

$$\frac{m}{r} = \frac{2-k}{5k-6}e^x r^{4(1-k)/(2-k)} + \frac{F_1(x)}{r}.$$

Formally solving this equation of the form

$$(\partial x/\partial r)_\tau = \theta(x;r), \quad (26)$$

we find

$$\tau = \bar{\tau}[\xi(x;r)], \quad (27)$$

where $\bar{\tau}(\xi)$ is an arbitrary function.

It is now easy to determine the quantity

$$e^{(\lambda-\nu)/2} = A = -\frac{\partial\tau}{\partial m}\frac{r^2(p+\varepsilon)}{4\Delta} = -\frac{\partial\tau}{\partial m}\frac{kpr^2}{4(k-1)\Delta} = \frac{\partial\tau}{\partial m}\frac{f(x)}{1-\varkappa m/r}. \tag{28}$$

Since $e^{\lambda} = 1/(1-\varkappa m/r)$, we find

$$e^{-\nu/2} = -\frac{\partial\tau}{\partial m}\frac{f(x)}{\sqrt{1-\varkappa m/r}}, \tag{29}$$

and thereby completely solve the problem posed.

Let us now turn to simplifications. For $\varkappa = 0$, i.e., in the absence of a gravitational field, we have the solution (2)

$$x = x(r-\tau), \qquad y = y(r-\tau), \qquad \sigma = \sigma(r-\tau), \tag{30}$$

which follows immediately from (7), (8), (9) for $A = 1$.

Neglecting in the second equation (13) the quantity $\Delta \ll 1$, we find that

$$\partial\tau/\partial r = A. \tag{31}$$

In this case equations (14), (15), (16) take the form

$$x = x(m), \qquad y = y(m), \qquad \sigma = \sigma(m), \tag{32}$$

$$(\partial m/\partial r)_\tau = f(x)/(1-\varkappa m/r) = f(m)/(1-\varkappa m/r). \tag{33}$$

For $\varkappa = 0$

$$(\partial m/\partial r)_\tau = f(m). \tag{34}$$

Integrating, we find that

$$r - \varphi(m) = T(\tau) = \tau,$$

whence $m = \psi(r-\tau)$ and $x = x(r-\tau)$, which gives the limiting results (30).

For $\varkappa \neq 0$ it is necessary to integrate equation (33):

$$\frac{dr}{1} = \frac{d\tau}{0}\frac{dm}{f(m)}\left(1-\frac{\varkappa m}{r}\right), \tag{35}$$

whence

$$\frac{dr}{dm} f(m) = 1 - \frac{\varkappa m}{r}.$$

Putting $r = 1/\xi$, we shall have

$$\frac{d\xi}{dm} f(m) = \varkappa m \xi^3 - \xi^2 \tag{36}$$

or

$$d\xi/d\eta = B(\eta)\xi^3 - \xi^2, \tag{37}$$

where $d\eta = dm/f(m)$; $B(\eta) = \varkappa m(\eta)$.

This equation is easily solved by numerical methods (for example, by the Runge-Kutta method); as a result, returning to the old variables, we find

$$D(r; m) = f(\tau), \tag{38}$$

which then permits one formally to express x, y, σ through r and τ.

In the case when $f = f_0 = \text{const}$, and this is an important case, equation (35) is integrated at once:

$$m^2 \left[\frac{r^2}{m^2} - \frac{r}{m f_0} + \frac{\varkappa}{f_0} \right] + F(\tau) \left[\frac{r/m - 1/2f_0 + (1/2f_0)\sqrt{1 - 4\varkappa f_0}}{-r/m + 1/2f_0 + (1/2f_0)\sqrt{1 - 4\varkappa f_0}} \right]^{1/\sqrt{1-4\varkappa f_0}}. \tag{39}$$

For $\chi = 0$

$$\frac{r}{m} f_0 = \frac{\psi(\tau)}{m} + 1; \tag{40}$$

further, from (39) we have

$$-\frac{m^2}{f_0^2} \left[\frac{\psi^2}{m^2} + \frac{\psi}{m} \right] - F(\tau) \left[\frac{\psi/m + 1}{\psi/m} \right] = 0;$$

whence

$$\frac{m^2}{f_0^2} \frac{\psi^2}{m^2} = F(\tau) = \frac{\psi^2(\tau)}{f_0^2},$$

which verifies the calculations we have carried out. From (20) it follows that in the case

$$f_1 = r_0^{4(k-1)/(2-k)} = \text{const}$$

with

$$\chi m/r = 1, \qquad r = r_g(m), \qquad \partial r/\partial m = 0 \quad (f(\chi) = f(m) \neq 0).$$

It also follows from this that here $r = r_{\text{min}} = 2Gm_0/c^2 = r_g$, where $m_0 = 4\pi m/c^2$, $\Delta = 0$, i.e., at the gravitational radius $u = c$. As $r \to \infty$, also $\Delta = 0$, $u = c$. The maximum value $\Delta = \Delta_{\text{max}}$, i.e., $u = u_{\text{min}}$, is attained at

$$r = \frac{3k-2}{4(k-1)}\chi m = \frac{3k-2}{4(k-1)}\frac{2Gm_0}{c^2} = \frac{3k-2}{4(k-1)}r_g.$$

Thus, in the ultrarelativistic limiting case there is a radius $r = r^*$ at which the velocity is minimal; this, however, still does not mean that the contraction of the medium stops.

Received
4 X 1968

CITED LITERATURE

1. K. P. Stanyukovich, DAN, 182, No. 2 (1968). 2. K. P. Stanyukovich, ZhETF, 36, No. 6 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.