Reports of the Academy of Sciences of the USSR

MECHANICS

1969

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1969. Volume 184, No. 5

UDC 531.36

MECHANICS

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STABILITY OF SECOND-ORDER DYNAMICAL SYSTEMS

(Presented by Academician I. I. Artobolevskii, 7 VI 1968)

The article sets forth a new method for the qualitative investigation of second-order dynamical systems, making it possible to study the question of the stability of equilibrium states of these systems and to establish the possibility of the existence of nonlinear oscillations and self-oscillations in the system.

Let the motion of a physical system be described by two nonlinear differential equations with respect to the variables x_1 and x_2

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2), \end{aligned} \tag{1}$$

where $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ are continuous and differentiable functions of the variables x_1 and x_2.

The singular points, or points of rest, of system (1), i.e., the equilibrium states of the dynamical system, are defined as the points $(x_1^{(p)}, x_2^{(p)})$ at which

$$f_i(x_1^{(p)}, x_2^{(p)}) = 0, \qquad i = 1, 2.$$

The study of the behavior of integral curves in the neighborhood of a singular point is of great importance both for questions of stability ([1,2]) and for the general qualitative analysis of differential equations ([3,4]).

Represent the functions f_1 and f_2 in the form

$$\begin{aligned} f_1(x_1, x_2) &= a_{11}x_1 + a_{12}x_2, \\ f_2(x_1, x_2) &= a_{21}x_1 + a_{22}x_2, \end{aligned} \tag{2}$$

where $a_{kj} = a_{kj}(x_1, x_2)$, $k = 1, 2$; $j = 1, 2$, are continuous and differentiable functions of the variables x_1 and x_2.

Introduce new variables by means of the equalities

$$\eta_s = C_{s1}x_1 + C_{s2}x_2, \qquad s = 1, 2, \tag{3}$$

where $C_{sk} = C_{sk}(x_1, x_2)$, $s = 1, 2$; $k = 1, 2$, are continuous and differentiable functions of the variables x_1 and x_2.

We require that equations (1) have, in the new variables, the canonical form:

$$\dot{\eta}_s = \rho_s \eta_s, \qquad s = 1, 2, \tag{4}$$

where ρ_s are certain functions.

Let the functions $C_{sk}(x_1, x_2)$ satisfy differential equations of the form

$$\dot{C}_{s1}x_1 + \dot{C}_{s2}x_2 = 0, \qquad s = 1, 2. \tag{5}$$

Substituting expressions (3) into equations (4) and taking into account equations (1) under condition (2), as well as equations (5), we obtain the equations

$$\begin{aligned} &(a_{11} - \rho_s)C_{s1} + a_{21}C_{s2} = 0, \\ &a_{12}C_{s1} + (a_{22} - \rho_s)C_{s2} = 0, \qquad s = 1, 2. \end{aligned} \tag{6}$$

The system of equations (6) has a nontrivial solution under the condition that

ρ_s are roots of the equation

$$\begin{vmatrix} a_{11} - \rho_s & a_{21} \\ a_{12} & a_{22} - \rho_s \end{vmatrix} = 0. \tag{7}$$

Thus, the functions $\rho_s = \rho_s(x_1, x_2)$, $s = 1, 2$, which may be called characteristic functions, are completely determined by the right-hand sides of equations (1), represented in the form (2).

Let us establish the conditions under which the relations (3) represent a one-to-one continuous transformation of the coordinate system Ox_1x_2 into the coordinate system $O\eta_1\eta_2$.

Solving equations (5) and (6) jointly, we obtain

$$C_{s2} = \varphi_s C_{s1}, \qquad C_{s1} = C_{s1}^{(0)} \exp\left[-\int_0^t \frac{\varphi_s' x_2}{x_1 + \varphi_s x_2}\, dt\right], \qquad s = 1, 2; \tag{8}$$

where

$$\varphi_s = a_{12}/(\rho_s - a_{22}) = (\rho_s - a_{11})/a_{21}, \qquad s = 1, 2; \tag{9}$$

$$C_{s1}^{(0)} = C_{s1}(x_{01}, x_{02}), \qquad s = 1, 2, \tag{10}$$

and the prime denotes differentiation with respect to time t.

The transformation (3) will take the form

$$\eta_s = C_{s1}^{(0)}(x_1 + \varphi_s x_2) \exp\left[-\int_0^t \frac{\varphi_s' x_2}{x_1 + \varphi_s x_2}\, dt\right], \qquad s = 1, 2. \tag{11}$$

Taking into account the obvious equality

$$x_1 + \varphi_s x_2 = (x_{01} + \varphi_s^{(0)} x_{02}) \exp\left[\int_{A_0 A} \frac{d(x_1 + \varphi_s x_2)}{x_1 + \varphi_s x_2}\right], \qquad s = 1, 2,$$

we represent the transformation (11) in the form

$$\eta_s = \eta_s^{(0)} \exp\left[\int_{A_0 A} Q_{1s}(x_1, x_2)\, dx_1 + Q_{2s}(x_1, x_2)\, dx_2\right], \qquad s = 1, 2, \tag{12}$$

where it is denoted that

$$Q_{1s}(x_1, x_2) = 1/(x_1 + \varphi_s x_2), \qquad Q_{2s}(x_1, x_2) = \varphi_s/(x_1 + \varphi_s x_2), \qquad s = 1, 2. \tag{13}$$

The variables η_1 and η_2 will be single-valued functions of the variables x_1 and x_2 provided that the curvilinear integral

$$\int_{(\Gamma)} Q_{1s}(x_1, x_2)\, dx_1 + Q_{2s}(x_1, x_2)\, dx_2$$

along a closed contour Γ in some simply connected domain is equal to zero, i.e., under the condition that

$$\partial Q_{1s}/\partial x_2 = \partial Q_{2s}/\partial x_1, \qquad s = 1, 2. \tag{14}$$

Conditions (14) are satisfied if the functions $\varphi_s(x_1, x_2)$ satisfy the relations

$$x_1 \partial\varphi_s/\partial x_1 + x_2 \partial\varphi_s/\partial x_2 = 0, \qquad s = 1, 2. \tag{15}$$

The transformation (12) will be nonsingular if the functional determinant (Jacobian) of this transformation is different from zero ([5]), i.e.,

$$\frac{D(\eta_1, \eta_2)}{D(x_1, x_2)} \neq 0. \tag{16}$$

If the functions $\eta_1(x_1, x_2)$ and $\eta_2(x_1, x_2)$ and their partial derivatives with respect to x_1

and x_2 are continuous in some domain L, and the Jacobian

$$\frac{D(\eta_1, \eta_2)}{D(x_1, x_2)}$$

is a continuous function of x_1 and x_2 in this domain E.

Computing the Jacobian of the transformation (12), we obtain that condition (16) is satisfied if $\varphi_1 \neq \varphi_2$, i.e., if the characteristic functions ρ_s satisfy the condition

$$\rho_1(x_1, x_2) \neq \rho_2(x_1, x_2). \tag{17}$$

Condition (16) is violated at those points of the plane at which the characteristic functions take the same value, i.e.,

$$\rho_1(x_1^*, x_2^*) = \rho_2(x_1^*, x_2^*).$$

The solution of the system of differential equations (4) can be represented in the form

$$\eta_s = \eta_s^{(0)} \exp\left[\int_0^t \rho_s(x_1, x_2)\, dt\right], \qquad s = 1, 2. \tag{18}$$

Consequently, questions concerning the stability of equilibrium states of the dynamical system, i.e., the character of the singular points of equations (1), the existence of periodic solutions of these equations, the existence and stability of limit cycles, etc., are decided on the basis of an investigation of the behavior of the characteristic functions $\rho_s(x_1, x_2)$ entering into solution (18) and determined from the characteristic functional equation (7).

For convenience, denote

$$I_s = \int_0^t \operatorname{Re}\rho_s(x_1, x_2)\, dt, \qquad s = 1, 2. \tag{19}$$

Thus, on the basis of the foregoing and of the mean-value theorem, one may assert that:

I. If $\operatorname{Re}\rho_s(x_1, x_2) < 0$, $s = 1, 2$, in a domain E containing the equilibrium position $(x_1^{(p)}, x_2^{(p)})$, then $I_s < 0$, $s = 1, 2$, for $t > 0$, and the equilibrium state of the dynamical system is asymptotically stable in the sense of Lyapunov.

II. If $\operatorname{Re}\rho_s(x_1, x_2) < 0$, $s = 1, 2$, at every point of the plane Ox_1x_2, then $I_s < 0$, $s = 1, 2$, for $t > 0$, and the equilibrium state of the dynamical system is asymptotically stable under arbitrary initial perturbations.

III. If $\operatorname{Re}\rho_s(x_1, x_2) > 0$ for at least one value of s in some domain E containing the equilibrium position $(x_1^{(p)}, x_2^{(p)})$, then $I_s > 0$ for $t > 0$, and the equilibrium state of the dynamical system is unstable in the sense of Lyapunov.

IV. If $\operatorname{Re}\rho_s = 0$, $\operatorname{Im}\rho_s \neq 0$ at every point of the domain E containing the equilibrium position $(x_1^{(p)}, x_2^{(p)})$, then $I_s = 0$, $s = 1, 2$, for $t > 0$, and the equilibrium state of the dynamical system is non-asymptotically stable in the sense of Lyapunov; in this case the domain E is entirely filled with closed trajectories corresponding to a non-isolated periodic solution.

V. If $\operatorname{Re}\rho_s > 0 (< 0)$, $s = 1, 2$, in a domain E containing the equilibrium position $(x_1^{(p)}, x_2^{(p)})$, while in a domain N containing the domain E inside it, $\operatorname{Re}\rho_s < 0 (> 0)$, and, consequently, $\operatorname{Re}\rho_s = 0$ on the boundary Γ of these domains, then for $t > 0$, $I_s > 0 (< 0)$, $s = 1, 2$, in the domain E, $I_s < 0 (> 0)$, $s = 1, 2$, in the domain N, $I_s = 0$, $s = 1, 2$, on Γ, and the curve $\Gamma_1(x_1, x_2)$, intersecting the curve Γ or coinciding with it, is a stable (unstable) limit cycle, i.e., there exists an isolated stable (unstable) periodic solution of system (1).

VI. If $\operatorname{Re}\rho_s > 0\,(< 0)$, $s = 1, 2$, in the domain E containing the equilibrium position $(x_1^{(p)}, x_2^{(p)})$, and $\operatorname{Re}\rho_s > 0\,(< 0)$, $s = 1, 2$, in the domain N, contained inside the domain E, while on the boundary Γ of these domains $\operatorname{Re}\rho_s = 0$, $s = 1, 2$, then, for $t > 0$, $I_s > 0\,(< 0)$, $s = 1, 2$, in the domain E, $I_s > 0\,(< 0)$, $s = 1, 2$, in the domain N, $I_s = 0$, $s = 1, 2$, on Γ, and the curve $\Gamma_1(x_1, x_2)$, intersecting the curve Γ or coinciding with it, is a semistable limit cycle.

VII. If $\operatorname{Re}\rho_s > 0\,(< 0)$, $s = 1, 2$, for $\bar{x}_1 < x_1 < \bar{\bar{x}}_1$ and $\operatorname{Re}\rho_s < 0\,(> 0)$, $s = 1, 2$, for $x_1 < \bar{x}_1$ and $x_1 > \bar{\bar{x}}_1$, and if no integral curve passes through the equilibrium position $(x_1^{(p)}, x_2^{(p)})$ for which $x_1^{(p)}$ lies inside the interval $(\bar{x}_1, \bar{\bar{x}}_1)$, then there exists a stable (unstable) limit cycle encompassing the interval $[\bar{x}_1, \bar{\bar{x}}_1]$.

Example. Suppose the motion of a physical system is described by the system of differential equations

$$\dot{x}_1 = a\left(x_1^2 + x_2^2 - 1\right)^k x_1 + \beta x_2, \\ \dot{x}_2 = \gamma x_1 + a\left(x_1^2 + x_2^2 - 1\right)^k x_2, \tag{20}$$

where a, β, γ, k are constants, with k an integer. The characteristic functions of the system of equations (20) have the form

$$\rho_{1,2} = a\left(x_1^2 + x_2^2 - 1\right)^k \pm \sqrt{\gamma\beta}.$$

Consequently:

a) if $k = 1, 3, ...,$ $\gamma\beta < 0$, then a neighborhood of the singular point $(0, 0)$ is of the type stable focus for $a > 0$, center for $a = 0$, and unstable focus for $a < 0$; system (20) has one limit cycle ($x_1^2 + x_2^2 = 1$, for $\gamma = -\beta$), stable for $a < 0$ and unstable for $a > 0$;

b) if $k = 2, 4, ...,$ $\gamma\beta < 0$, then a neighborhood of the singular point $(0, 0)$ is of the type stable focus for $a < 0$, center for $a = 0$, and unstable focus for $a > 0$; system (20) has one semistable limit cycle ($x_1^2 + x_2^2 = 1$, for $\gamma = -\beta$);

c) if $k = 1, 3, ...,$ $\gamma\beta > 0$, then a neighborhood of the singular point $(0, 0)$ is of the type unstable node for $a < -\sqrt{\gamma\beta}$, saddle for $-\sqrt{\gamma\beta} < a < \sqrt{\gamma\beta}$, and stable node for $a > \sqrt{\gamma\beta}$;

d) if $k = 2, 4, ...,$ $\gamma\beta > 0$, then a neighborhood of the singular point $(0, 0)$ is of the type stable node for $a < -\sqrt{\gamma\beta}$, saddle for $-\sqrt{\gamma\beta} < a < \sqrt{\gamma\beta}$, and unstable node for $a > \sqrt{\gamma\beta}$.

Received
4 VI 1968

REFERENCES

1. A. M. Lyapunov, *The General Problem of the Stability of Motion*, Moscow-Leningrad, 1950.

2. I. G. Malkin, *Theory of Stability of Motion*, Nauka, 1966.

3. V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, Moscow-Leningrad, 1949.

4. A. A. Andronov, E. A. Leontovich et al., *Qualitative Theory of Second-Order Dynamical Systems*, Nauka, 1966.

5. G. M. Fichtenholz, *Course of Differential and Integral Calculus*, 3rd ed., Moscow–Leningrad, 1951.

Note: Figure translations are in progress. See original paper for figures.